

Does a Conformal Transformation Change the Quasinormal Mode Spectrum in Weyl Gravity?

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Introduction

- Quasinormal modes (QNMs) probe the **dynamics** of a black hole, not its line element alone.
- Conformally related black holes have repeatedly been reported to possess different QNM spectra.
- Existing Weyl/conformal-gravity studies span test fields, axial perturbations, and ringdown phenomenology [[1807.01792](#), [1902.01678](#), [1905.12290](#), [1910.00428](#), [2002.06360](#), [2012.13020](#), [2501.16134](#), [2503.16087](#)].
- However, these calculations do not determine the **complete linearized Bach gravitational problem**; a conformal metric alone does not specify its perturbation dynamics.
- We construct the complete axial sector of pure Weyl gravity and determine when the **full QNM spectrum** is conformally invariant.

Previous results

- Consider a regular conformal Schwarzschild geometry:

$$ds^2 = C^2(r) ds_{\text{Schw}}^2, \quad C(r) = \left(1 + \frac{L^2}{r^2}\right)^N. \quad (1)$$

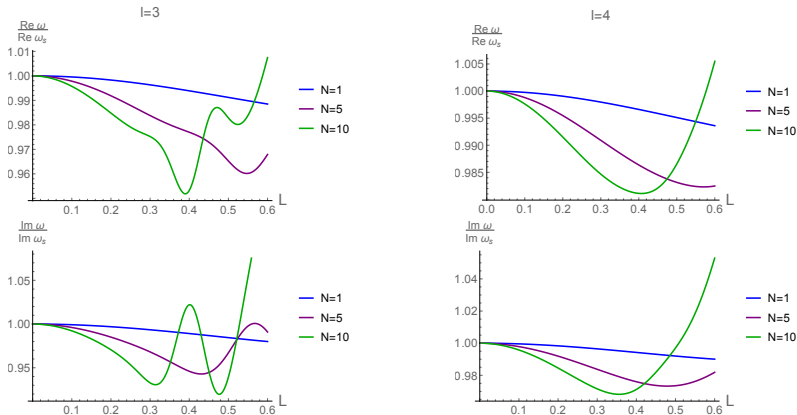
- In the earlier treatment, this metric was assigned an effective anisotropic-fluid source in **Einstein gravity**, rather than pure Weyl gravity.
- The axial perturbation then obeys a single second-order equation:

$$\frac{d^2 H^{(-)}}{dr_*^2} + [\omega^2 - V_g(r)] H^{(-)} = 0. \quad (2)$$

- Here $Z = rC$ is the areal-radius factor of the conformal geometry, so the resulting frequencies vary with L and N .
- $H^{(-)}$ is the axial master variable of this effective Einstein problem.

QNM shifts

- Sixth-order WKB frequencies are shown for $\ell = 3, 4$ and $N = 1, 5, 10$.
- Both the oscillation frequency and damping rate deviate from their Schwarzschild values; the deviation is stronger at larger N [1902.01678].



The issue

- The preceding shifts follow from an Einstein equation with an **effective anisotropic-fluid energy-momentum tensor**.
- Thus the conformal factor changes both the metric and the equation selected for the perturbation.
- Local conformal covariance of the background is therefore insufficient to establish QNM isospectrality.
- We must retain **all gravitational degrees of freedom** in the full conformally invariant theory.

Compensated Einstein gravity

- As a transparent comparison, consider Einstein gravity with a compensator scalar:

$$S_{\text{CE}} = \int d^4x \sqrt{-g} [\phi^2 R + 6(\nabla\phi)^2], \quad g_{\mu\nu} \rightarrow C^2 g_{\mu\nu}, \quad \phi \rightarrow C^{-1}\phi. \quad (3)$$

- The metric and compensator transform together under $g_{\mu\nu} \rightarrow C^2 g_{\mu\nu}$ and $\phi \rightarrow C^{-1}\phi$.
- The invariant metric is

$$\gamma_{\mu\nu} = \left(\frac{\phi}{\phi_0}\right)^2 g_{\mu\nu}, \quad S_{\text{CE}} = \phi_0^2 \int d^4x \sqrt{-\gamma} R[\gamma]. \quad (4)$$

- The physical dynamics is simply Einstein gravity for $\gamma_{\mu\nu}$.
- This example separates a conformal frame change from a change of physical dynamics.

Invariant axial equation

- For a conformal Schwarzschild background, transform the compensator together with the metric:

$$g_{\mu\nu} = C^2(r)\hat{g}_{\mu\nu}, \quad \phi = C^{-1}(r)\phi_0, \quad Q(r) = \frac{f(r)h_1(r)}{rC^2(r)}. \quad (5)$$

- $Q = fh_1/(rC^2)$ is the invariant axial master variable. The combination Q is **invariant** under this simultaneous transformation.
- It obeys the Regge-Wheeler equation:

$$\frac{d^2Q}{dr_*^2} + [\omega^2 - V_2(r)] Q = 0, \quad V_2(r) = f(r) \left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3} \right]. \quad (6)$$

- Hence the Schwarzschild spectrum is recovered when the boundary conditions are unchanged.

Pure Weyl gravity

- The main problem is pure Weyl gravity:

$$S_W = \int d^4x \sqrt{-g} \mathcal{C}_{\mu\nu\rho\sigma} \mathcal{C}^{\mu\nu\rho\sigma}, \quad B_{\mu\nu} = 0. \quad (7)$$

$\mathcal{C}_{\mu\nu\rho\sigma}$ is the Weyl tensor and $B_{\mu\nu}$ is the Bach tensor.

- Its Bach equation is fourth order and contains non-Einstein gravitational degrees of freedom.
- On an Einstein seed, introduce the linearized Einstein tensor:

$$E_{\mu\nu}[h] \equiv \delta G_{\mu\nu}[h] - \hat{\Lambda} h_{\mu\nu}. \quad (8)$$

- $E_{\mu\nu}$ preserves the non-Einstein sector while reducing the radial problem to second order.
- This is the variable from which we reconstruct the full axial metric perturbation.

Master variables

- Axial harmonic separation starts from amplitudes (Ψ, Ξ) ; the master variable Q is constructed from them and the Regge-Wheeler operator $\mathcal{R}_2$.
- Thus Q is the non-Einstein tensor master variable, while Ξ carries the companion axial amplitude.
- Their radial equations contain a Maxwell-type operator $\mathcal{R}_1$ and a Regge-Wheeler operator $\mathcal{R}_2$:

$$\mathcal{R}_1 = \partial_{r_*}^2 + \omega^2 - F \frac{\ell(\ell+1)}{\rho^2}, \quad \mathcal{R}_2 = \partial_{r_*}^2 + \omega^2 - F \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{6M}{\rho^3} \right], \quad (9)$$

where F is the Schwarzschild-(A)dS shape function.

- Therefore $\mathcal{R}_2$ is the tensor Regge-Wheeler operator, while $\mathcal{R}_1$ has the axial Maxwell form.

Coupled master equations

- The complete axial Bach problem is a coupled second-order system for $(\Xi, Q)^\top$:

$$\begin{pmatrix} \mathcal{R}_1 & -2F \\ -\frac{8M\hat{\Lambda}F}{3\rho^3} & \mathcal{R}_2 - \frac{2\hat{\Lambda}F}{3} \end{pmatrix} \begin{pmatrix} \Xi \\ Q \end{pmatrix} = 0. \quad (10)$$

- The off-diagonal terms carry the physical mixing between the two axial channels. This mixing is proportional to $\hat{\Lambda}$ in the lower-left entry.
- For $\hat{\Lambda} = 0$, the system becomes triangular and its spectral branches can be separated explicitly.
- The variables (Ψ, Ξ, Q) reconstruct the complete axial metric perturbation rather than a selected second-order subsector.

Schwarzschild spectrum

- For Schwarzschild, $\hat{\Lambda} = 0$ and Q is the non-Einstein tensor master variable in a triangular system:

$$\mathcal{R}_1 \Xi = 2FQ, \quad \mathcal{R}_2 Q = 0. \quad (11)$$

- A first-order radial operator $\mathcal{D}$, constructed from M , ρ , ℓ , ω , and ∂_ρ , removes the off-shell source term.

- Then $Z_1 = \Xi - \mathcal{D}Q$ removes the source term; Z_1 is the decoupled spin-1 master variable:

$$\mathcal{R}_2 Q = 0, \quad \mathcal{R}_1 Z_1 = 0. \quad (12)$$

- The two equations give the Regge-Wheeler and spin-1 branches:

$$\text{Spec}_{\text{Bach}} = \text{Spec}_{\text{RW}} \cup \text{Spec}_{s=1}. \quad (13)$$

- Regge-Wheeler frequencies are second-order poles; spin-1 frequencies are simple poles.

Conformal map of master variables

- Let $\tilde{g}_{\mu\nu} = C^2(\rho)g_{\mu\nu}^{\text{Sch}}$, with the same time and tortoise coordinates. Write $\mathcal{P} = (\Xi, Q)^\top$ and $q = C'/C$, where the prime denotes $d/d\rho$.

- The master-vector transformation is local and triangular:

$$\tilde{\mathcal{P}} = \mathcal{T}_C \mathcal{P}, \quad \mathcal{T}_C = C^2 \begin{pmatrix} 1 & 0 \\ 2\rho q & 1 \end{pmatrix}, \quad q = \frac{C'}{C}. \quad (14)$$

- The radial operators obey the relation

$$\tilde{\mathcal{L}}_B(\omega) \mathcal{T}_C = \mathcal{L}_B(\omega). \quad (15)$$

- If C is regular, $\mathcal{T}_C$ and $\mathcal{T}_C^{-1}$ map local solutions one-to-one.
- QNM invariance additionally requires that the boundary conditions are mapped one-to-one.

When is the spectrum preserved?

- A QNM obeys ingoing and outgoing boundary conditions:

$$u \sim e^{-i\omega r_*} \quad (\text{horizon}), \quad u \sim e^{+i\omega r_*} \quad (\text{spatial infinity}). \quad (16)$$

- First, C and C^{-1} must be regular at both relevant endpoints.
- The boundaries must map one-to-one and the time normalization must be fixed:

$$\text{Spec}_{\text{QNM}}(\tilde{\mathcal{L}}_{\text{B}}) = \text{Spec}_{\text{QNM}}(\mathcal{L}_{\text{B}}). \quad (17)$$

- The conformal map then preserves the incoming/outgoing solution spaces.
- Hence frequencies, multiplicities, and pole orders are preserved.

Numerical check

- We test a regular conformal factor:

$$C^2(\rho) = \left(1 + \frac{L^2}{\rho^2}\right)^2, \quad L = M = 1. \quad (18)$$

- Direct Leaver calculations and transformed shooting calculations are compared.
- We test both the Regge-Wheeler and spin-1 branches, including an overtone:

	n=0	n=5
$ \Delta\omega_{\text{RW}} $	4.35×10^{-14}	1.26×10^{-12}
$ \Delta\omega_{s=1} $	6.46×10^{-14}	3.23×10^{-11}

- The residuals are at the numerical precision of the two computations.
- The agreement verifies regular conformal isospectrality of the complete axial spectrum.

Summary

- Schwarzschild Bach perturbations contain both the Regge-Wheeler and spin-1 branches.
- The Regge-Wheeler frequencies are **second-order poles** of the full Bach resolvent, while the spin-1 poles are simple.
- A regular conformal transformation preserves the **complete QNM spectrum**, including multiplicities and pole orders.
- A singular transformation can change the endpoint domain and therefore the boundary-value problem and spectrum.
- The method extends to polar Bach perturbations (after $\delta R = 0$ gauge fixing) and to higher-derivative gravities, including quadratic-curvature Schwarzschild-(A)dS and Einstein-Ricci cubic Schwarzschild-AdS backgrounds.

Thank you for your attention!