

3D Summation-By-Parts Scheme on Hyperboloidal Slices

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Motivation

- ▶ Ultimate aim:
 - ▶ Evolving any Initial Data (ID), containing any matter distribution, using the EFEs and the matter equations to any desired accuracy.
 - ▶ Extract the resulting Gravitational Waves (GW) and accompanying signal at future null infinity, $\mathcal{I}^+$.
- ▶ Why $\mathcal{I}^+$?
 - ▶ The only region where Gravitational Waves (GWs) can be defined unambiguously.
 - ▶ Useful for studying phenomena of fundamental interest, like weak cosmic censorship conjecture, black hole formation, etc.
- ▶ Needs to be addressed geometrically, analytically and numerically:

A Solution: Formulating EFEs on Hyperboloidal Slices

- ▶ Geometrical Part:
 - ▶ Foliating the 4D spacetime via 3D hypersurfaces.
 - ▶ One Solution: Use hyperboloidal slices¹.
- ▶ Analytical Part:
 - ▶ Defining a Well-Posed formulation of the EFEs².
 - ▶ One Solution: Use Generalized Harmonic Gauge (GHG)³ together with Dual-Foliation Formalism⁴.
- ▶ Numerical Part:
 - ▶ Deriving discretization schemes preserving the continuum well-posed structure at the discrete level.
 - ▶ One Solution: Use Summation-by-Parts (SBP) scheme⁵.

¹Friedrich et al 1983-86, Zenginoğlu et al 2005-Present and many more!

²Fourès-Bruhat 1952, Lindblad and Rodnianski 2003 and many more!

³Friedrich 1985, Gasperín and Hilditch 2018, Duarte et al 2021-22.

⁴Hilditch et al 2015-16

⁵Gundlach et al 2010, Gautam et al 2021, and many more!

Hyperboloidal Slices vs Cauchy Slices

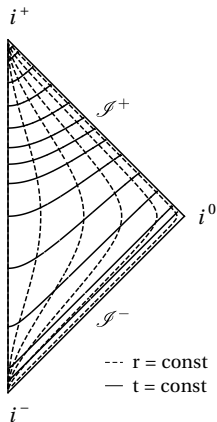


Figure: Hyperboloidal slices in Minkowski spacetime.

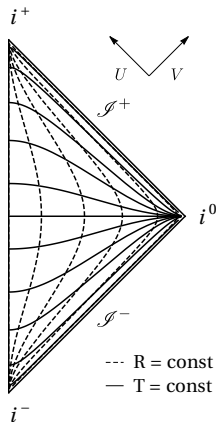


Figure: Cauchy Slices in Minkowski spacetime.

Harmonic Gauge

- ▶ Vacuum EFEs, $R_{ab} = 0$

$$g^{cd} \partial_c \partial_d g_{ab} = 2 \nabla_{(a} \Gamma_{b)} + 2 g^{cd} g^{ef} (\partial_e g_{ac} \partial_f g_{bd} - \Gamma_{ace} \Gamma_{bdf}).$$

- ▶ $\Gamma_{abc} \equiv \frac{1}{2} (\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc})$, $\Gamma_a \equiv g^{bc} \Gamma_{abc}$.

- ▶ *Harmonic gauge*: $\Gamma^a = 0$. This name is derived from the fact that $\square x^a = -\Gamma^a$. $\therefore \Gamma^a = 0 \Rightarrow \square x^a = 0$.

- ▶ Gives a hyperbolic form:

$$g^{cd} \partial_c \partial_d g_{ab} = 2 g^{cd} g^{ef} (\partial_e g_{ac} \partial_f g_{bd} - \Gamma_{ace} \Gamma_{bdf}).$$

- ▶ Constraint: $C^a \equiv \Gamma^a = 0$.

Generalized Harmonic Gauge

- ▶ *Generalized Harmonic gauge*: $\square x^a = F^a$ or $-\Gamma^a = F^a$.
 F^a : a function of x^a and g_{ab} , but not its derivatives.

- ▶ Gives another hyperbolic form:

$$g^{cd}\partial_c\partial_d g_{ab} = -2\nabla_{(a}F_{b)} + 2g^{cd}g^{ef}(\partial_e g_{ac}\partial_f g_{bd} - \Gamma_{ace}\Gamma_{bdf}).$$

- ▶ Inhomogenous equations. F^a : Gauge source function.
- ▶ GHG constraint: $C^a \equiv F^a + \Gamma^a = 0$.
- ▶ Vacuum EFE in GHG:

$$R_{ab} - \nabla_{(a}C_{b)} = 0.$$

Dual-Foliation Formalism

- ▶ Consider two spacetime foliations $X^\mu = (T, X^i)$ and $x^\mu = (t, x^i)$, via spacelike hypersurfaces.
- ▶ Introduce 3 + 1 split of the Jacobian $J^\mu{}_\mu = dX^\mu/dx^\mu$

$$\partial_{\underline{T}} = (J^{-1})^\mu{}_{\underline{T}} \partial_\mu, \quad \partial_{\underline{i}} = (J^{-1})^\mu{}_{\underline{i}} \partial_\mu.$$

- ▶ Dynamical variables: $u_i(T, R) \rightarrow u_i(t, r)$ remain the same.
- ▶ An equation for a state vector $\mathbf{u} = \{u_1, u_2, \dots, u_n\}$ of the form

$$\partial_T \mathbf{u} = \mathbf{A}^p \partial_p \mathbf{u} + A \mathbf{S},$$

becomes

$$\partial_t \mathbf{u} = \mathbf{B}^p \partial_p \mathbf{u} + B \mathbf{S}.$$

Summation-by-Parts (SBP) scheme: Overall idea⁶

- ▶ A discrete version of Integration-By-Parts
 - ▶ $\Rightarrow$ Change in f = flux at the boundaries.

$$\int_a^b Df = f_b - f_a.$$

- ▶ Assures stability when discretized properly.
- ▶ Applying on conservative systems:
 - ▶ For $\nabla_\mu J^\mu = 0$, define $\mathcal{H} \equiv n_\mu J^\mu$, and $J_i \equiv \gamma_{i\mu} J^\mu$, rewrite as $\partial_t \mathcal{H} = -D_i J^i$.
 - ▶ Integral form

$$\dot{E}(t) = \int_{\mathcal{R}} \partial_t \mathcal{H} = - \int_{\mathcal{R}} D_i J^i = - \int_{\partial \mathcal{R}} s_i J^i,$$

s^i = unit normal to ∂R .

⁶Strand, 1993

Summation-by-Parts (SBP) scheme: Overall idea⁷

- ▶ Discretize D_i such that, at the discrete level, one obtains

$$\sum_{\mathcal{R}} [D_i] [J^i] = \sum_{\partial\mathcal{R}} [s_i] [J^i]$$

- ▶ For a scalar field ψ satisfying the linear wave equation (LWE):

$$(\square - F)\psi = 0,$$

conserved energy: obtained from the stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}(\partial^\alpha\psi\partial_\alpha\psi + F\psi^2),$$

- ▶ In the case of Einstein Field Equations, one needs to work with quantities analogous to ADM or Bondi energies etc.
- ▶ Project on hyperboloidal slices Σ_t .

⁷Strand, 1993

SBP scheme: Spacetime Region ⁸

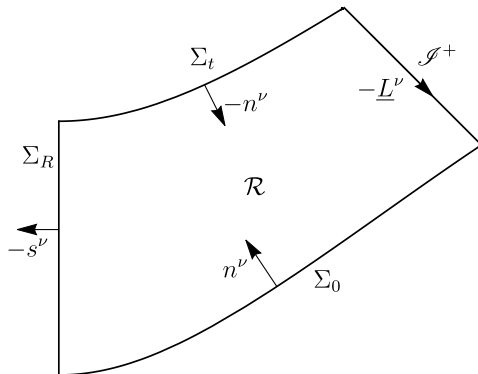


Figure: Spacetime region under consideration

⁸SG, A. Vañó-Viñuales, D. Hilditch, S. Bose, PRD 103, 084045

Linear Wave Equation

- ▶ In Spherical polar coordinates: Everywhere

$$\left[-\partial_T^2 + \frac{1}{R^2} \partial_R \left(R^2 \partial_R \right) + \frac{1}{R^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{R^2 \sin^2 \theta} \partial_\phi^2 - F \right] \psi = 0,$$

- ▶ On the z-axis:

$$\left[-\partial_T^2 + \frac{1}{R^2} \partial_R \left(R^2 \partial_R \right) + \frac{2}{R^2} \partial_\theta^2 - F_{(m=0)} \right] \psi_{(m=0)} = 0,$$

- ▶ At the origin:

$$\left[-\partial_T^2 + 3 \partial_R^2 - F_{(l=0)} \right] \psi_{(l=0)} = 0,$$

- ▶ Here, the $m = 0$ modes are extracted from:

$$\psi_{(m=0)}(T, R, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \psi(T, R, \theta, \phi) d\phi,$$

Linear Wave Equation

- ▶ And the $l = 0$ mode can be extracted from

$$\psi_{(l=0)}(T, R) = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \psi(T, R, \theta, \phi) \sin \theta \, d\theta \, d\phi.$$

- ▶ The functions $F_{(m=0)}$ and $F_{(l=0)}$ are defined similarly.
- ▶ Introduce First-Order Reduction (FOR): $\psi_{\alpha} \equiv \partial_{\alpha} \psi$

$$\begin{aligned} \psi_T &\equiv \partial_T \psi, & \psi_R &\equiv \partial_R \psi, & \psi_{\theta} &\equiv \partial_{\theta} \psi, & \psi_{\phi} &\equiv \partial_{\phi} \psi, \\ \psi_+ &\equiv \psi_T + \psi_R, & \psi_- &\equiv \psi_T - \psi_R. \end{aligned}$$

- ▶ Rescaling:

$$\begin{aligned} \tilde{\psi} &\equiv \chi \psi, & \tilde{\psi}_+ &\equiv \chi_+ \psi_+, & \tilde{\psi}_- &\equiv \chi_- \psi_-, & \tilde{\psi}_{\theta} &\equiv \chi \psi_{\theta}, & \tilde{\psi}_{\phi} &\equiv \chi \psi_{\phi}, \\ \chi_+ &= \chi^2, & \chi_- &= \chi & \text{for } r > 0, & \chi(0) &= 1, & \chi &\sim R \text{ for } R \gg 1. \end{aligned}$$

- ▶ Regularized covariant divergence:

$$\tilde{\partial}_r f \equiv \frac{\chi^2}{R^2} \partial_r \left(\frac{R^2}{\chi^2} f \right), \quad \tilde{\partial}_{\theta} f \equiv \frac{1}{\sin \theta} \partial_{\theta} ((\sin \theta) f), \quad \tilde{\partial}_{\phi} \equiv \partial_{\phi}.$$

Linear Wave Equation

► Discrete EOMs:

$$\dot{\tilde{\Psi}} = \frac{1}{2} \left(\left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + \tilde{\Psi}_- \right),$$

$$\begin{aligned} \dot{\tilde{\Psi}}_+ = & \left[\frac{\chi}{2R' - 1} \right] \left((\Upsilon^r)^{-1} \left[\frac{D_r + \tilde{D}_r}{2} \right] \left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + (\Upsilon^r)^{-1} \left[\frac{D_r - \tilde{D}_r}{2} \right] \tilde{\Psi}_- - \left[\frac{\chi'}{\chi} \right] \tilde{\Psi}_- \right. \\ & + \left[\frac{R'}{R^2} \right] \left((\Upsilon^\theta)^{-1} \tilde{D}_\theta \tilde{\Psi}_\theta + \left[\frac{1}{\sin^2 \theta} \right] (\Upsilon^\phi)^{-1} \tilde{D}_\phi \tilde{\Psi}_\phi \right) - [R'F] \tilde{\Psi} + \zeta \left[\frac{1}{\chi} \right] \left((\Upsilon^r)^{-1} D_r \tilde{\Psi} \right. \\ & \left. \left. - \left[\frac{\chi'}{\chi} \right] \tilde{\Psi} - \left[\frac{2R' - 1}{2\chi} \right] \tilde{\Psi}_+ + \frac{\tilde{\Psi}_-}{2} \right) \right), \end{aligned}$$

$$\begin{aligned} \dot{\tilde{\Psi}}_- = & -(\Upsilon^r)^{-1} \left[\frac{D_r + \tilde{D}_r}{2} \right] \tilde{\Psi}_- - (\Upsilon^r)^{-1} \left[\frac{D_r - \tilde{D}_r}{2} \right] \left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + \left[\frac{R'}{R^2} \right] \left((\Upsilon^\theta)^{-1} \tilde{D}_\theta \tilde{\Psi}_\theta \right. \\ & + \left[\frac{1}{\sin^2 \theta} \right] (\Upsilon^\phi)^{-1} \tilde{D}_\phi \tilde{\Psi}_\phi \left. \right) + \left[\frac{\chi'}{\chi^2} \right] \tilde{\Psi}_+ - [R'F] \tilde{\Psi} - \zeta \left[\frac{1}{\chi} \right] \left((\Upsilon^r)^{-1} D_r \tilde{\Psi} - \left[\frac{\chi'}{\chi} \right] \tilde{\Psi} \right. \\ & \left. \left. - \left[\frac{2R' - 1}{2\chi} \right] \tilde{\Psi}_+ + \frac{\tilde{\Psi}_-}{2} \right) \right), \end{aligned}$$

$$\dot{\tilde{\Psi}}_\theta = \frac{1}{2} (\Upsilon^\theta)^{-1} D_\theta \left(\left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + \tilde{\Psi}_- \right) + \zeta \left[\frac{1}{\chi} \right] \left((\Upsilon^\theta)^{-1} D_\theta \tilde{\Psi} - \tilde{\Psi}_\theta \right),$$

$$\dot{\tilde{\Psi}}_\phi = \frac{1}{2} (\Upsilon^\phi)^{-1} D_\phi \left(\left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + \tilde{\Psi}_- \right) + \zeta \left[\frac{1}{\chi} \right] \left((\Upsilon^\phi)^{-1} D_\phi \tilde{\Psi} - \tilde{\Psi}_\phi \right),$$

Linear Wave Equation

- Conserved Energy:

$$E(t) \equiv \frac{1}{2} \int_{\Sigma_t} \left(F R' \tilde{\psi}^2 + \left(\frac{2R' - 1}{2\chi^2} \right) \tilde{\psi}_+^2 + \frac{\tilde{\psi}_-^2}{2} + \frac{R'}{R^2} \left(\tilde{\psi}_\theta^2 + \frac{1}{\sin^2 \theta} \tilde{\psi}_\phi^2 \right) \right) \frac{R^2}{\chi^2} \sin \theta \, dr \, d\theta \, d\phi$$

- Energy flux:

$$\begin{aligned} \partial_t E(t, r_o(t)) &\equiv \int_{r=0}^{r_o(t)} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \partial_t \epsilon \, dr \, d\theta \, d\phi \\ &= \frac{1}{4} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(\frac{\tilde{\psi}_+^2}{\chi^2} - \tilde{\psi}_-^2 \right) \frac{R^2}{\chi^2} \sin \theta \, d\theta \, d\phi \Big|_{r=r_o}, \end{aligned}$$

- Flux at $\mathcal{I}^+$

$$\frac{d}{dt} E(t) = - \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{4} \left(F \tilde{\psi}^2 + \tilde{\psi}_-^2 \right) \sin \theta \, d\theta \, d\phi \Big|_{r=r_{\mathcal{I}^+}} \leq 0.$$

Linear Wave Equation

- ▶ Ansatz: Discrete Energy:

$$\hat{E}(t) = \frac{1}{2} \left(\tilde{\Psi}^T \Upsilon [F] \tilde{W} \tilde{\Psi} + \tilde{\Psi}_+^T \Upsilon \tilde{W}_+ \tilde{\Psi}_+ + \tilde{\Psi}_-^T \Upsilon \tilde{W}_- \tilde{\Psi}_- + \tilde{\Psi}_\theta^T \Upsilon \tilde{W}_\theta \tilde{\Psi}_\theta + \tilde{\Psi}_\phi^T \Upsilon \tilde{W}_\phi \tilde{\Psi}_\phi \right),$$

- ▶ With Weight Matrices:

$$\tilde{W} = \left[\frac{R' R^2 \sin \theta}{\chi^2} \right], \quad \tilde{W}_+ = \left[\frac{(2R' - 1) R^2 \sin \theta}{2 \chi^4} \right], \quad \tilde{W}_- = \left[\frac{R^2 \sin \theta}{2 \chi^2} \right],$$
$$\tilde{W}_\theta = \left[\frac{R' \sin \theta}{\chi^2} \right], \quad \tilde{W}_\phi = \left[\frac{R'}{\chi^2 \sin \theta} \right].$$

- ▶ Discrete Energy Flux:

$$\partial_t \hat{E}(t) = \tilde{\Psi}_+^T \left[\frac{1}{\chi} \right] \Upsilon^\theta \Upsilon^\phi B \left[\frac{1}{\chi} \right] \tilde{\Psi}_+ - \tilde{\Psi}_-^T \Upsilon^\theta \Upsilon^\phi B \tilde{\Psi}_-,$$

SBP Scheme

Time derivative and substituting discrete EOMs:

$$\begin{aligned}\partial_t \hat{E}(t) = & \tilde{\Psi}_+^T \left[\frac{1}{\chi} \right] \Upsilon^\theta \Upsilon^\phi \tilde{W}_- \left[\frac{D_r + \tilde{D}_r}{2} \right] \left[\frac{1}{\chi} \right] \tilde{\Psi}_+ - \tilde{\Psi}_-^T \Upsilon^\theta \Upsilon^\phi \tilde{W}_- \left[\frac{D_r + \tilde{D}_r}{2} \right] \tilde{\Psi}_- \\ & + \tilde{\Psi}_+^T \left[\frac{1}{\chi} \right] \Upsilon^\theta \Upsilon^\phi \tilde{W}_- \left[\frac{D_r - \tilde{D}_r}{2} \right] \tilde{\Psi}_- - \tilde{\Psi}_-^T \Upsilon^\theta \Upsilon^\phi \tilde{W}_- \left[\frac{D_r - \tilde{D}_r}{2} \right] \left[\frac{1}{\chi} \right] \tilde{\Psi}_+ \\ & + \tilde{\Psi}_+^T \left[\frac{1}{\chi} \right] \Upsilon^r \Upsilon^\phi \frac{\tilde{W}_\theta}{2} \tilde{D}_\theta \tilde{\Psi}_\theta + \tilde{\Psi}_\theta^T \Upsilon^r \Upsilon^\phi \frac{\tilde{W}_\theta}{2} D_\theta \left[\frac{1}{\chi} \right] \tilde{\Psi}_+ \\ & + \tilde{\Psi}_-^T \Upsilon^r \Upsilon^\phi \frac{\tilde{W}_\theta}{2} \tilde{D}_\theta \tilde{\Psi}_\theta + \tilde{\Psi}_\theta^T \Upsilon^r \Upsilon^\phi \frac{\tilde{W}_\theta}{2} D_\theta \tilde{\Psi}_- + \tilde{\Psi}_+^T \left[\frac{1}{\chi} \right] \Upsilon^r \Upsilon^\theta \frac{\tilde{W}_\phi}{2} \tilde{D}_\phi \tilde{\Psi}_\phi \\ & + \tilde{\Psi}_\phi^T \Upsilon^r \Upsilon^\phi \frac{\tilde{W}_\phi}{2} D_\phi \left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + \tilde{\Psi}_-^T \Upsilon^r \Upsilon^\theta \frac{\tilde{W}_\phi}{2} \tilde{D}_\phi \tilde{\Psi}_\phi + \tilde{\Psi}_\phi^T \Upsilon^r \Upsilon^\phi \frac{\tilde{W}_\phi}{2} D_\phi \tilde{\Psi}_- .\end{aligned}$$

SBP Scheme

- We get the SBP scheme:

$$\begin{aligned}\tilde{W}_- \left[\frac{D_r + \tilde{D}_r}{2} \right] + \left(\tilde{W}_- \left[\frac{D_r + \tilde{D}_r}{2} \right] \right)^T &= 2B, \\ \tilde{W}_- \left[\frac{D_r - \tilde{D}_r}{2} \right] - \left(\tilde{W}_- \left[\frac{D_r - \tilde{D}_r}{2} \right] \right)^T &= 0, \\ \tilde{W}_\theta \tilde{D}_\theta + (\tilde{W}_\theta D_\theta)^T &= 0, \quad \tilde{W}_\phi \tilde{D}_\phi + (\tilde{W}_\phi D_\phi)^T,\end{aligned}$$

- This gives:

$$\begin{aligned}\tilde{D}_r &= -\tilde{W}_-^{-1} D_r^T \tilde{W}_- + 2 \tilde{W}_-^{-1} B, \\ \tilde{D}_\theta &= -\tilde{W}_\theta^{-1} D_\theta^T \tilde{W}_\theta, \quad \tilde{D}_\phi = -D_\phi^T.\end{aligned}$$

Fixed background: Spherical Symmetry (Work in Progress)

- Consider the metric

$$\mathbf{g} = \begin{pmatrix} \frac{C_+ C_-}{C_+ - C_-} & -\frac{C_+ + C_-}{C_+ - C_-} & 0 & 0 \\ -\frac{C_+ + C_-}{C_+ - C_-} & \frac{2}{C_+ - C_-} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 \sin^2 \theta \end{pmatrix},$$

where $C_{\pm} = C_{\pm}(R)$ are outgoing and incoming coordinate lightspeeds.

- Examples:

- Schwarzschild in Schwarzschild coordinates:
 $C_+ = -C_- = 1 - 2M/R$.
- Schwarzschild in Kerr-Schild coordinates: $C_- = -1$,
 $C_+ = (1 - 2M/R)/(1 + 2M/R)$.
- Reissner Nordstrom etc.

- The scheme remains the same as above!

Dissipation

- The original system is modified as:

$$\begin{aligned}\partial_T \psi &= \psi_T, \\ \partial_T \psi_T &= \frac{1}{R^2} \partial_R (R^2 \psi_R) + \frac{1}{R^2 \sin \theta} \partial_\theta (\sin \theta \psi_\theta) + \frac{1}{R^2 \sin^2 \theta} \partial_\phi \psi_\phi - F \psi + \mathbf{Q} \psi_T, \\ \partial_T \psi_R &= \partial_R \psi_T + \zeta (\partial_R \psi - \psi_R), \\ \partial_T \psi_\theta &= \partial_\theta \psi_T + \zeta (\partial_\theta \psi - \psi_\theta), \\ \partial_T \psi_\phi &= \partial_\phi \psi_T + \zeta (\partial_\phi \psi - \psi_\phi).\end{aligned}$$

- The discrete system is now modified as

$$\begin{aligned}\dot{\tilde{\Psi}}_+ &= \dots + \left[\frac{\chi}{2R' - 1} \right] \tilde{Q} \left(\left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + \tilde{\Psi}_- \right), \\ \dot{\tilde{\Psi}}_- &= \dots + \tilde{Q} \left(\left[\frac{1}{\chi} \right] \tilde{\Psi}_+ + \tilde{\Psi}_- \right),\end{aligned}$$

- Obtained from the powers of discrete version of a regularized Laplacian operator

$$\tilde{Q}^{(4)} \equiv \left(\tilde{\partial}_r \partial_r + \frac{\chi^2}{R^2} \left(\tilde{\partial}_\theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \right)^2,$$

Results: $F = 0$

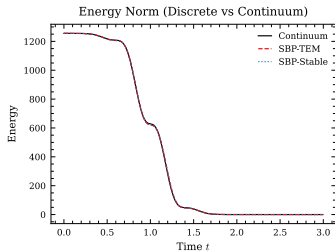


Figure: Discrete vs Continuum energy, $F = 0$.

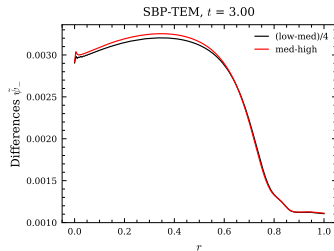


Figure: r -convergence, integrated over θ and ϕ .

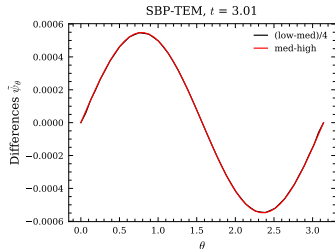


Figure: θ -convergence, integrated over r and ϕ .

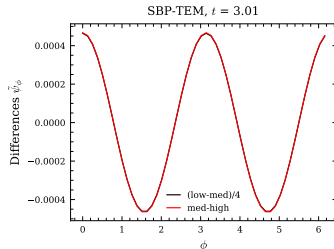


Figure: Energy with time, $F = m^2$.

Results: $F = 1/R^2, m^2$

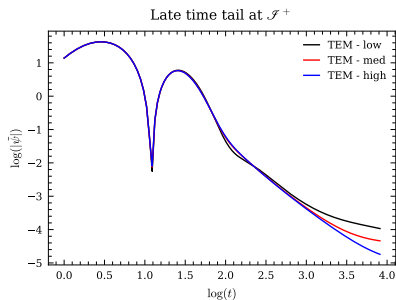


Figure: r -convergence, integrated over θ and ϕ .

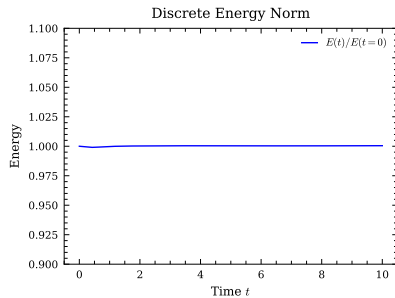


Figure: Energy with time, $F = m^2$.

Summary and Conclusions

- ▶ Unambiguous extraction of GW at $\mathcal{I}^+$ in NR requires solving the problem geometrically, analytically and numerically.
- ▶ Geometric part can be addressed by introducing hyperboloidal slices, and Analytic part can be solved by introducing GHG with DF formulation.
- ▶ Numerical part needs constructing an SBP scheme, which leads to stable codes by construction, essentially by demanding well-posedness at the discrete level.
- ▶ Overcomes coordinate singularities, and produces stable evolutions even for singular systems, like massive fields.
- ▶ Adding first-order reduction constraints and dissipation carries a geometrical meaning.
- ▶ Can also be extended to any conservation law $\nabla_\mu J^\mu = 0$ on a general and dynamical spacetimes $g_{ab}(x^\mu)$ (work in progress).

Thank You for your Attention!! Questions? Comments?

Existing method 1: Extrapolation

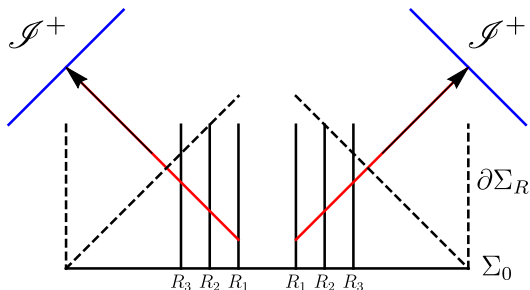


Figure: Extrapolation schematics.

Extrapolation Methods from Cauchy evolution:

- ▶ Systematic errors, depend on the specifics.
- ▶ Boundary errors: limit the evolution time.
- ▶ Fails to capture asymptotic charges.

Existing method 2: Cauchy Characteristic Extraction (CCE)⁹:

- ▶ Boundary errors: limit the evolution time.
- ▶ Gauge ambiguities at finite radial coordinate.
- ▶ Cauchy and characteristic regions decoupled: No backreaction effects.

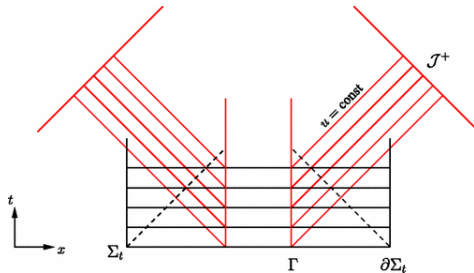


Figure: CCE schematics.

Source: C. Reisswig, N. T. Bishop, D. Pollney, and B. Szilágyi, Phys. Rev. Lett. 103, 221101 (2009).

⁹Bishop+ 1996, 2009, 2013, Taylor+ 2013, Handmer+ 2015-16 etc.

Proposed method: Cauchy Characteristic Matching (CCM)

Winicour 2009.

- ▶ Simultaneous Cauchy and Characteristic evolutions with data exchange.
- ▶ Gauge ambiguities.
- ▶ Successful evolutions for test cases (Ma et al 2023). Fully nonlinear case still unclear.

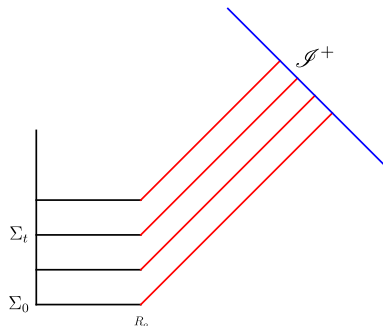


Figure: CCM schematics.