

Linear Stability of Dyonic Black Holes in $U(1)$ Gauge-Invariant Scalar-Vector-Tensor Theories

Shunta Nishimura

Tokyo University of Science

Contents

1. Introduction

Background, Previous Work, Motivation and Objectives

2. Black Hole Perturbation Analysis

Setup, Overview of Stability Analysis

3. Results and Summary

Contents

1. Introduction

Background, Previous Work, Motivation and Objectives

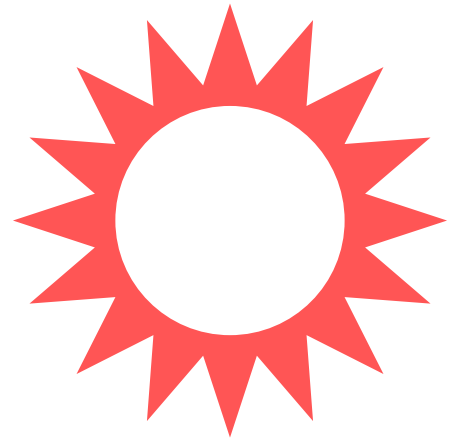
2. Black Hole Perturbation Analysis

Setup, Overview of Stability Analysis

3. Results and Summary

Introduction: No-Hair Theorem and Black Hole Hair

- No-hair theorem

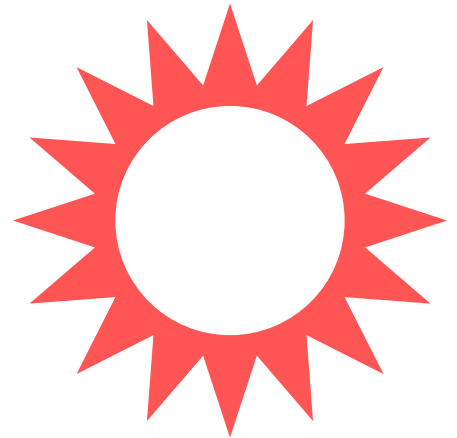


Introduction: No-Hair Theorem and Black Hole Hair

- No-hair theorem

$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory



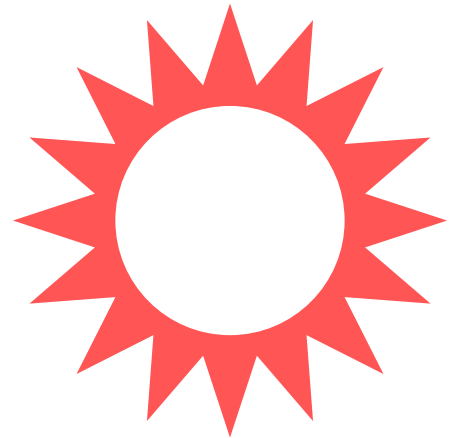
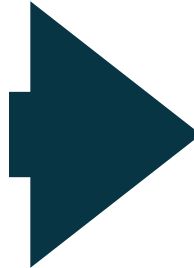
Introduction: No-Hair Theorem and Black Hole Hair

- No-hair theorem

$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

Gravitational Collapse



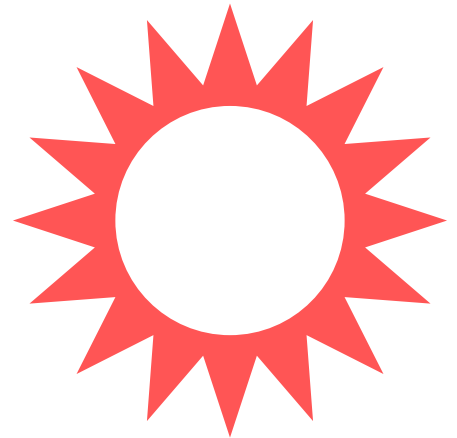
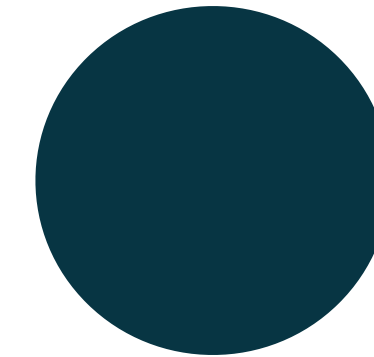
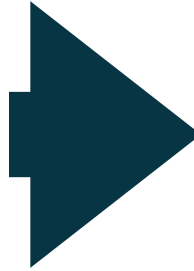
Introduction: No-Hair Theorem and Black Hole Hair

- No-hair theorem

$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

Gravitational Collapse



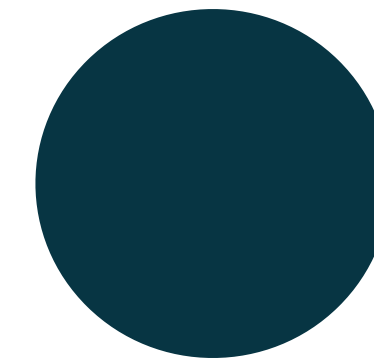
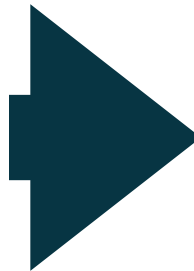
Introduction: No-Hair Theorem and Black Hole Hair

- No-hair theorem

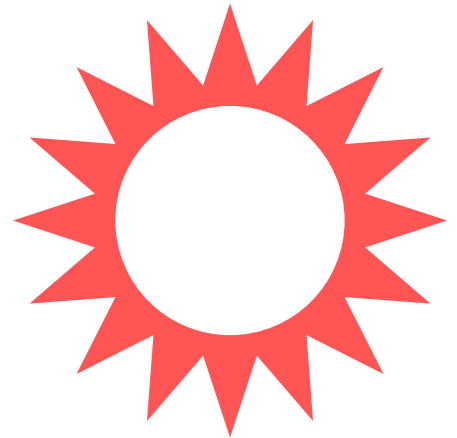
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

Gravitational Collapse

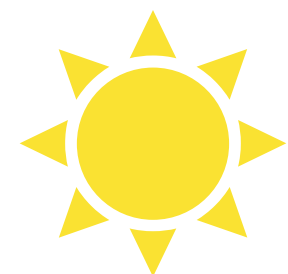
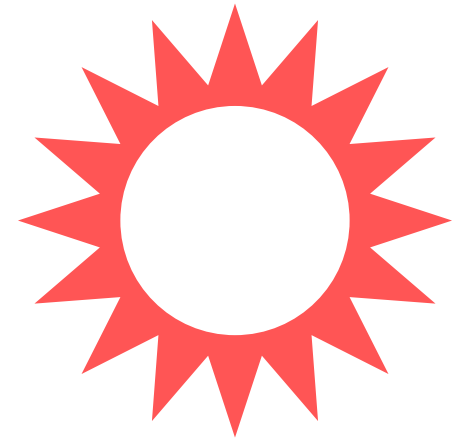


Mass, charge, and angular momentum only



Introduction: No-Hair Theorem and Black Hole Hair

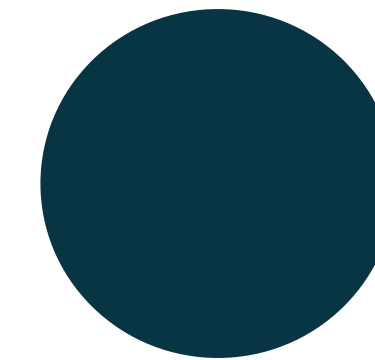
- No-hair theorem



$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

Gravitational Collapse



Mass, charge, and angular momentum only

$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R + L_{\text{EMS}}(\phi, F)$$

Einstein-Maxwell + Scalar Field

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Introduction: No-Hair Theorem and Black Hole Hair

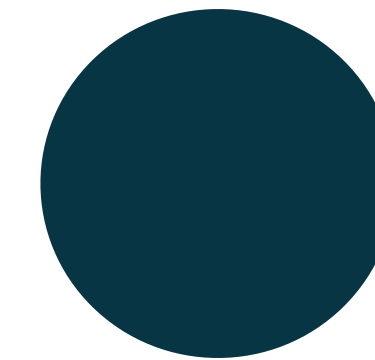
- No-hair theorem



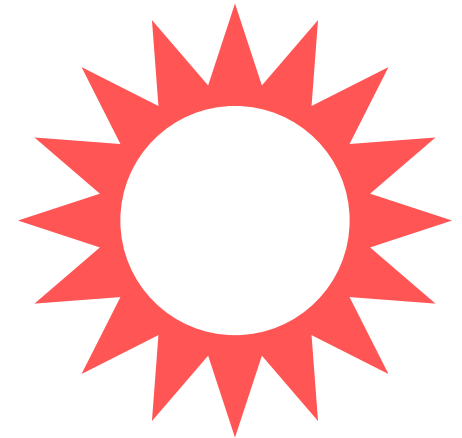
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

■ Gravitational Collapse ➡



Mass, charge, and angular momentum only

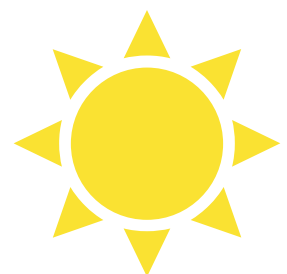


$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R + L_{\text{EMS}}(\phi, F)$$

Einstein-Maxwell + Scalar Field

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

■ Gravitational Collapse ➡



Introduction: No-Hair Theorem and Black Hole Hair

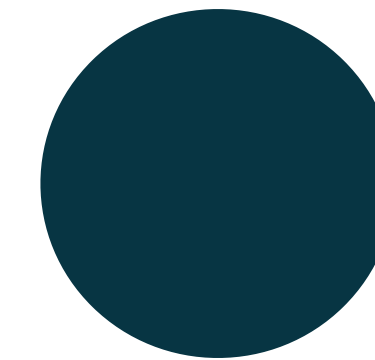
- No-hair theorem



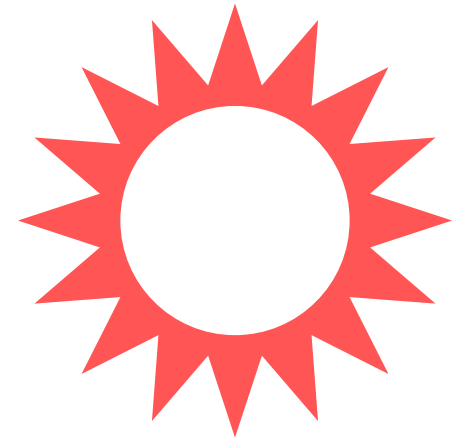
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

■ Gravitational Collapse ➡



Mass, charge, and angular momentum only



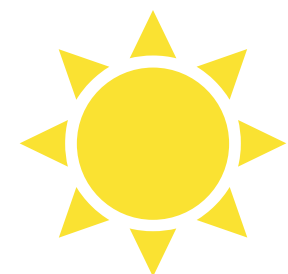
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R + L_{\text{EMS}}(\phi, F)$$

Einstein-Maxwell + Scalar Field

■ Gravitational Collapse ➡



$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$



Introduction: No-Hair Theorem and Black Hole Hair

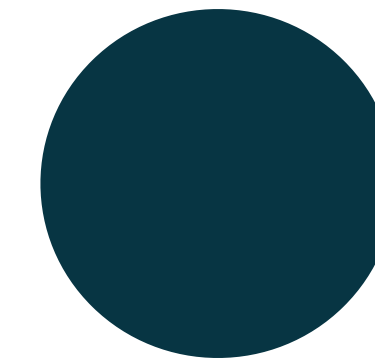
- No-hair theorem



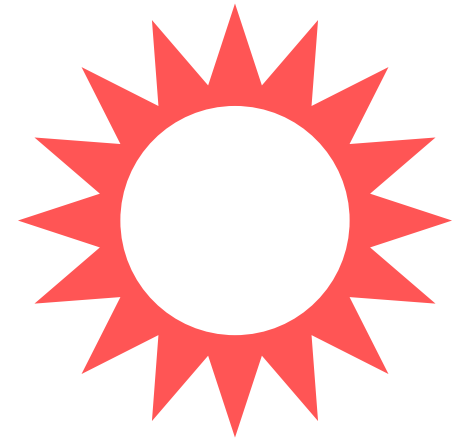
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

■ Gravitational Collapse ➡



Mass, charge, and angular momentum only



$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R + L_{\text{EMS}}(\phi, F)$$

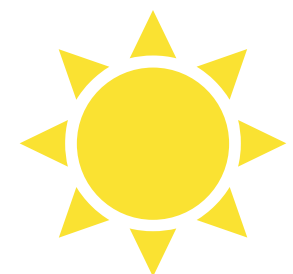
Einstein-Maxwell + Scalar Field

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

■ Gravitational Collapse ➡



M, Q, J + additional physical quantities



Introduction: No-Hair Theorem and Black Hole Hair

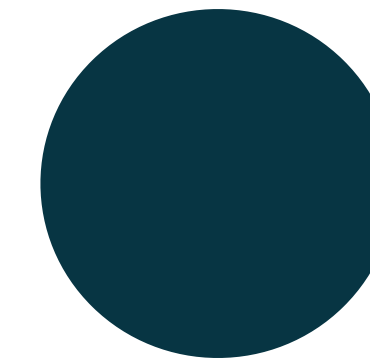
- No-hair theorem



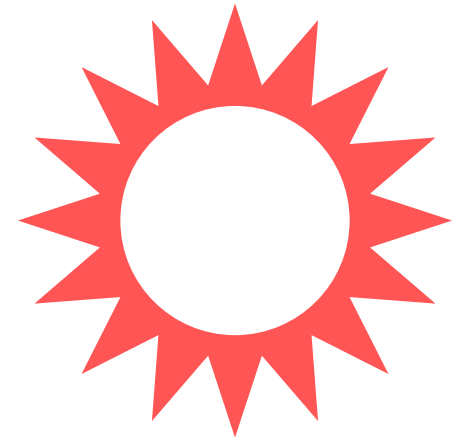
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

■ Gravitational Collapse ➡



Mass, charge, and angular momentum only



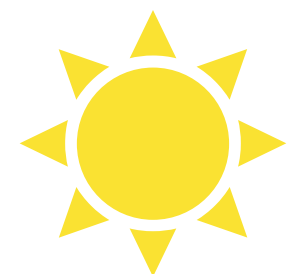
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R + L_{\text{EMS}}(\phi, F)$$

Einstein-Maxwell + Scalar Field

■ Gravitational Collapse ➡



M, Q, J + additional physical quantities



Introduction: No-Hair Theorem and Black Hole Hair

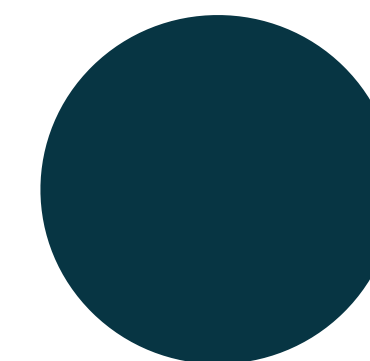
- No-hair theorem



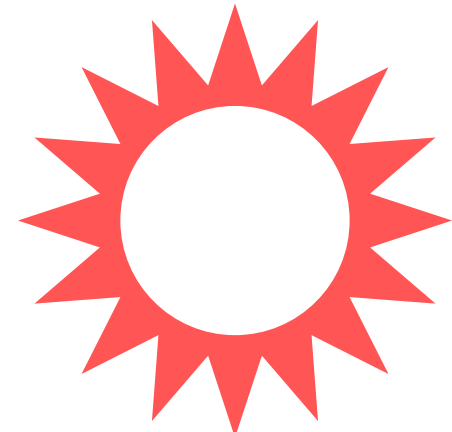
$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Einstein-Maxwell theory

Gravitational Collapse



Mass, charge, and angular momentum only



$$\mathcal{L} = \frac{M_{\text{pl}}^2}{2} R + L_{\text{EMS}}(\phi, F)$$

Einstein-Maxwell + Scalar Field

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

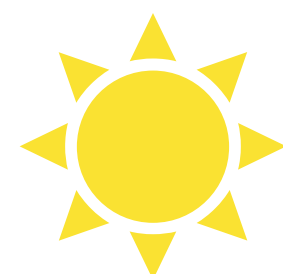
Gravitational Collapse



Hair

Hairy BH

M, Q, J + additional physical quantities



Introduction: No-Hair Theorem and Black Hole Hair

- No-hair theorem

Black hole hair can
leave imprints on observables.

→ Observing such signatures
would indicate physics beyond
Einstein-Maxwell theory.

Gravitational Collapse

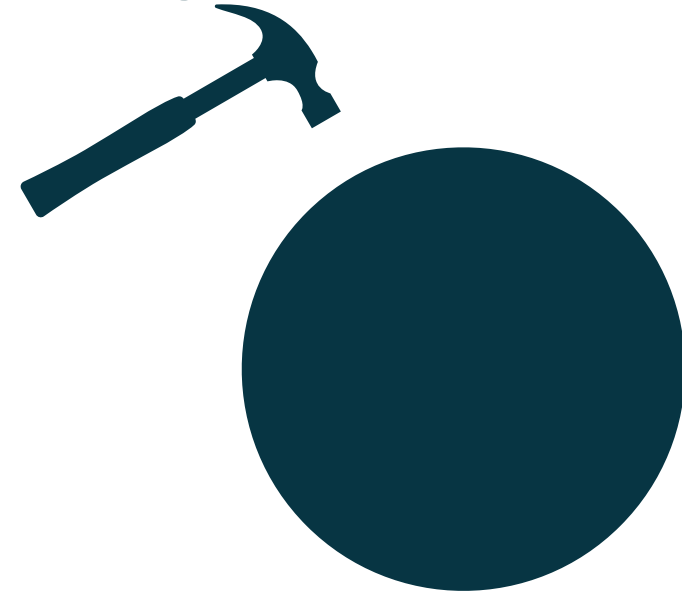
M, Q, J + additional physical quantities

Hair

Hairy BH

Introduction: Previous Work — Stability Analysis

- What is stability?

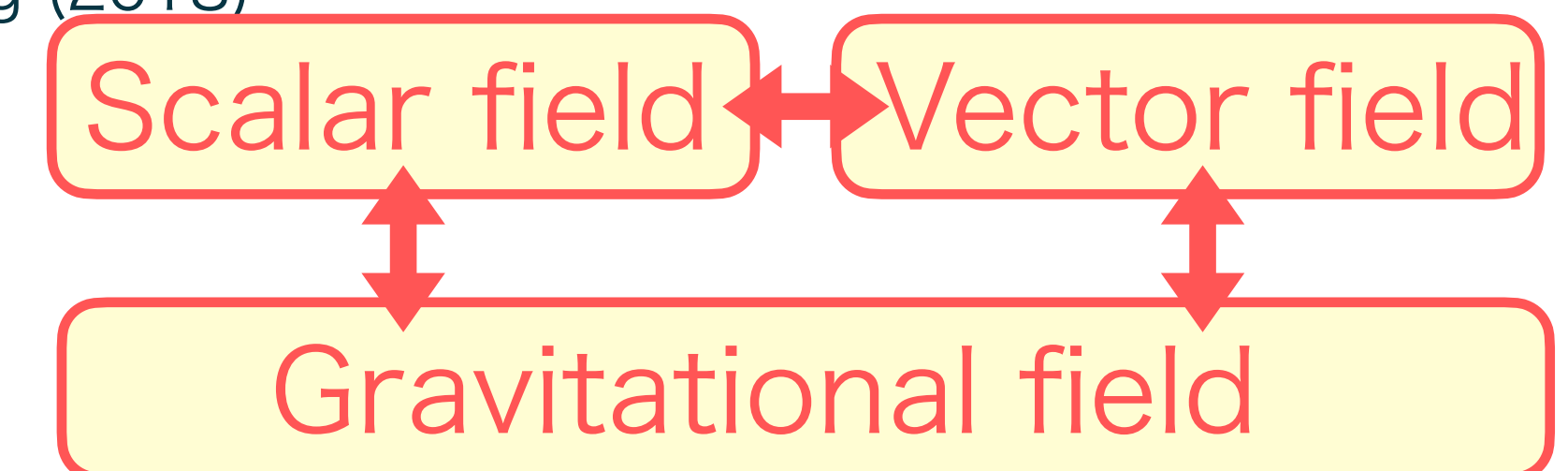


- Stability analysis of hairy BH solutions arising from scalar-vector-tensor couplings

- **Scalar-Vector-Tensor (SVT) theories**

L. Heisenberg (2018)

General scalar-vector-tensor interactions with second-order equations of motion



- **Stability of black holes in U(1) gauge-invariant SVT theories**
with electric charge only

L. Heisenberg, R. Kase and S. Tsujikawa (2018) ; C. Zhang and R. Kase (2024)

Introduction: Previous Work — Stability Analysis

- What is stability?

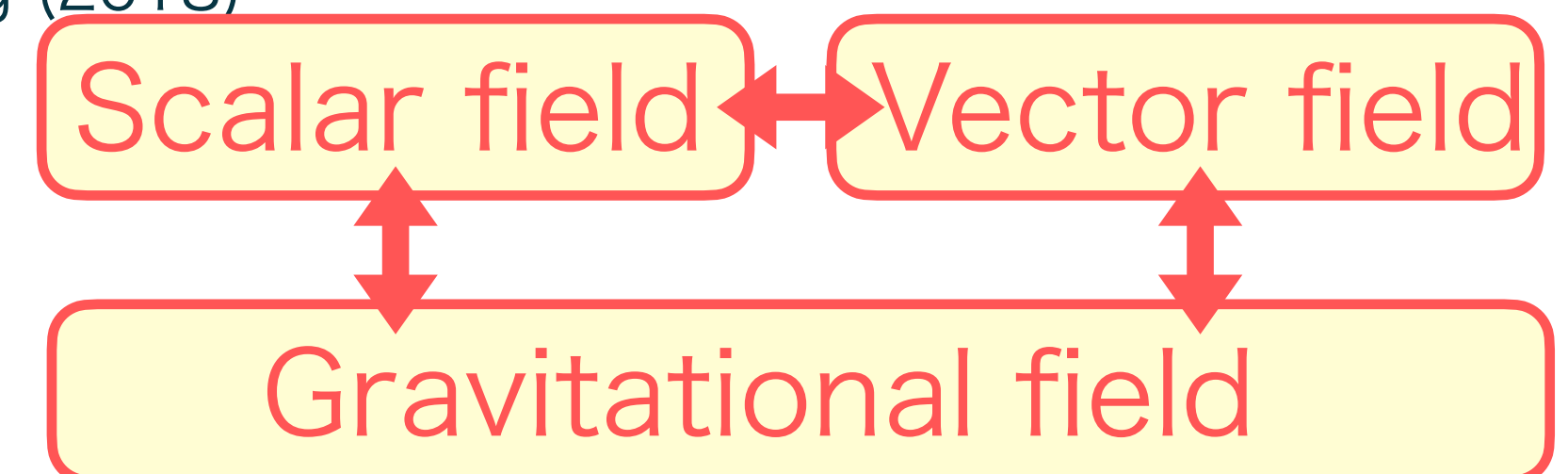


- Stability analysis of hairy BH solutions arising from scalar-vector-tensor couplings

- **Scalar-Vector-Tensor (SVT) theories**

L. Heisenberg (2018)

General scalar-vector-tensor interactions with second-order equations of motion

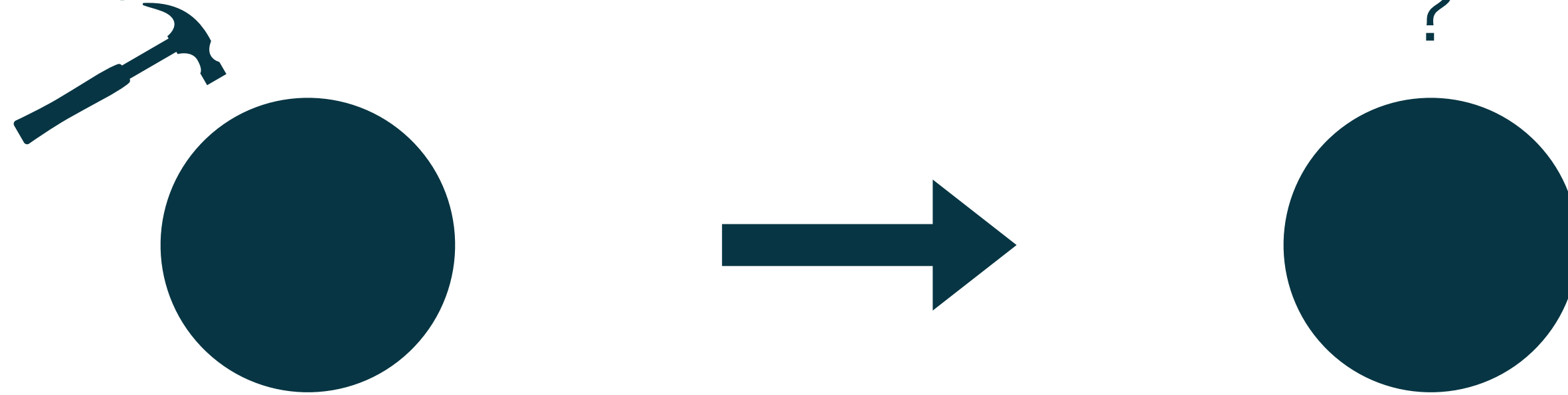


- **Stability of black holes in U(1) gauge-invariant SVT theories**
with electric charge only

L. Heisenberg, R. Kase and S. Tsujikawa (2018) ; C. Zhang and R. Kase (2024)

Introduction: Previous Work — Stability Analysis

- What is stability?

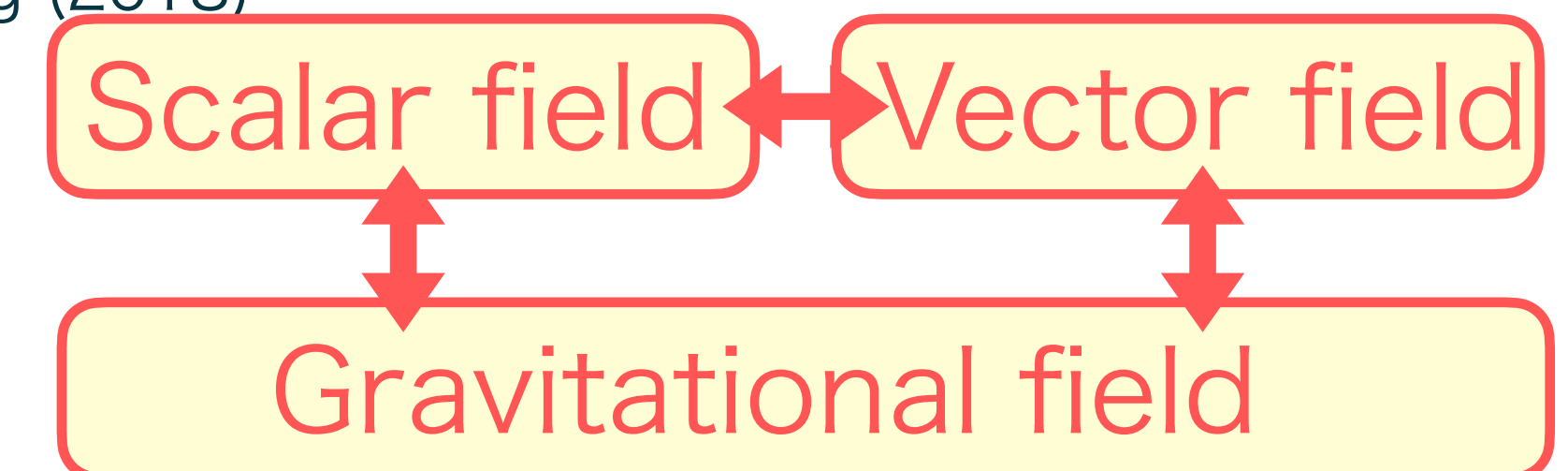


- Stability analysis of hairy BH solutions arising from scalar-vector-tensor couplings

- **Scalar-Vector-Tensor (SVT) theories**

L. Heisenberg (2018)

General scalar-vector-tensor interactions with second-order equations of motion

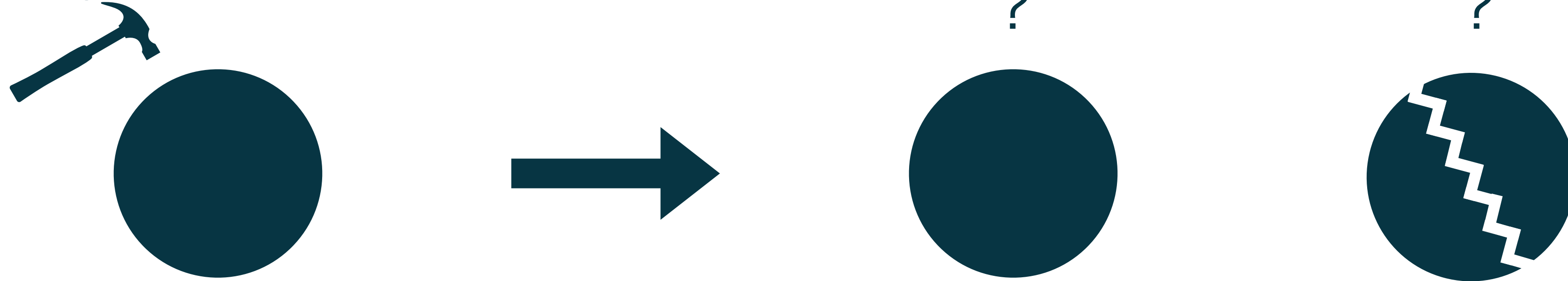


- **Stability of black holes in U(1) gauge-invariant SVT theories**
with electric charge only

L. Heisenberg, R. Kase and S. Tsujikawa (2018) ; C. Zhang and R. Kase (2024)

Introduction: Previous Work — Stability Analysis

- What is stability?

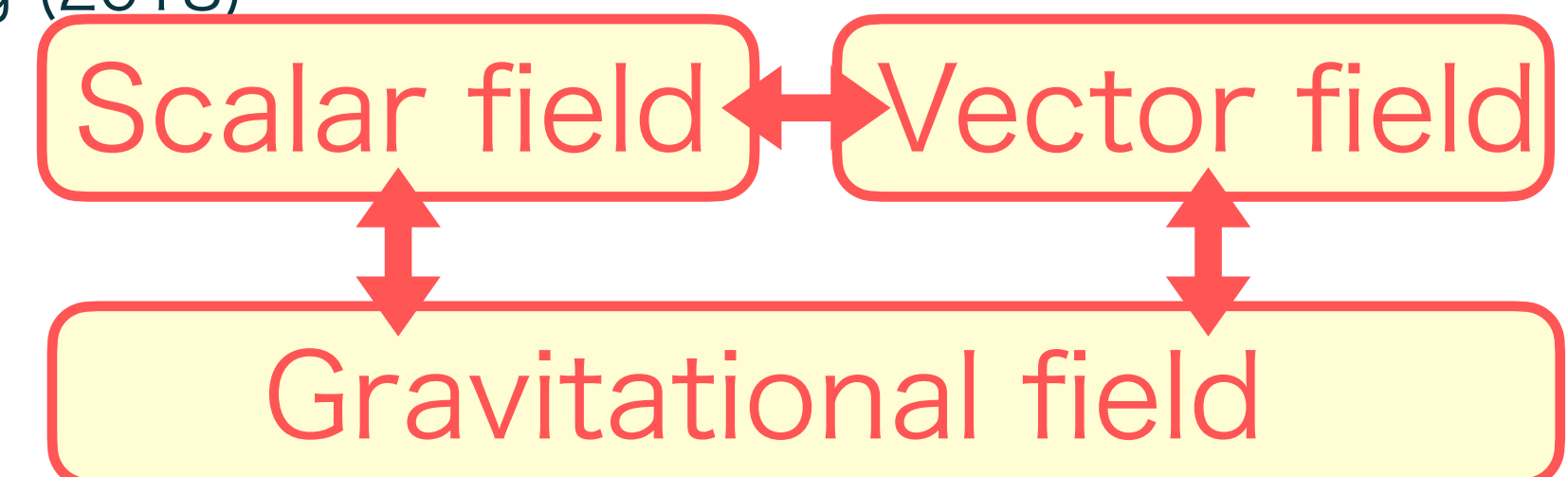


- Stability analysis of hairy BH solutions arising from scalar-vector-tensor couplings

- **Scalar-Vector-Tensor (SVT) theories**

L. Heisenberg (2018)

General scalar-vector-tensor interactions with second-order equations of motion



- **Stability of black holes in U(1) gauge-invariant SVT theories**
with electric charge only

L. Heisenberg, R. Kase and S. Tsujikawa (2018) ; C. Zhang and R. Kase (2024)

Introduction: Motivation and Objectives

- Previous studies considered **only electric charge**. However, ...
 - Hairy black hole solutions admitting **dyonic** configurations **with both electric and magnetic charges**
G. W. Gibbons and K.-i. Maeda (1988) ; D. Garfinkle, G. T. Horowitz, and A. Strominger (1991)
 - Hairy solutions that **arise only in the presence of magnetic charge**
K.-M. Lee and E. J. Weinberg (1991)
 - Recently discovered examples
K. Taniguchi, S. Takagishi and R. Kase (2024); M. Kitagawa, N. Tsukamoto and R. Kase (2026)

Introduction: Motivation and Objectives

- Previous studies considered **only electric charge**. However, ...
 - Hairy black hole solutions admitting **dyonic** configurations **with both electric and magnetic charges**
G. W. Gibbons and K.-i. Maeda (1988) ; D. Garfinkle, G. T. Horowitz, and A. Strominger (1991)
 - Hairy solutions that **arise only in the presence of magnetic charge**
K.-M. Lee and E. J. Weinberg (1991)
 - Recently discovered examples
K. Taniguchi, S. Takagishi and R. Kase (2024); M. Kitagawa, N. Tsukamoto and R. Kase (2026)

► **A general stability analysis including magnetic charge is therefore required.**

Introduction: Motivation and Objectives

- Previous studies considered **only electric charge**. However, ...
 - Hairy black hole solutions admitting **dyonic** configurations **with both electric and magnetic charges**

G. W. Gibbons and K.-i. Maeda (1988) ; D. Garfinkle, G. T. Horowitz, and A. Strominger (1991)

- Hairy solutions that **arise only in the presence of magnetic charge**

K.-M. Lee and E. J. Weinberg (1991)

- Recently discovered examples

K. Taniguchi, S. Takagishi and R. Kase (2024); M. Kitagawa, N. Tsukamoto and R. Kase (2026)

▶ **A general stability analysis including magnetic charge is therefore required.**

- Objectives of this work

1. Derive general stability conditions in U(1) gauge-invariant SVT theories.
2. Apply these conditions to specific models admitting **dyonic black hole solutions** and **investigate their stability**.

Contents

1. Introduction

Background, Previous Work, Motivation and Objectives

2. Black Hole Perturbation Analysis

Setup, Overview of Stability Analysis

3. Results and Summary

Setup

- Action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R \right]$$

Setup

- Action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R + \mathcal{L}_{\text{SVT}}^{(2)} + \mathcal{L}_{\text{SVT}}^{(3)} + \mathcal{L}_{\text{SVT}}^{(4)} \right]$$

$$\mathcal{L}_{\text{SVT}}^{(2)} = f_2(\phi, X, F, \tilde{F}, Y),$$

$$\mathcal{L}_{\text{SVT}}^{(3)} = \mathcal{M}_3^{\mu\nu} \nabla_\mu \nabla_\nu \phi,$$

$$\mathcal{M}_3^{\mu\nu} = \left[f_3(\phi, X) g_{\rho\sigma} + \tilde{f}_3(\phi, X) \nabla_\rho \phi \nabla_\sigma \phi \right] \tilde{F}^{\mu\rho} \tilde{F}^{\nu\sigma},$$

$$\mathcal{L}_{\text{SVT}}^{(4)} = \mathcal{M}_4^{\mu\nu\alpha\beta} \nabla_\mu \nabla_\alpha \phi \nabla_\nu \nabla_\beta \phi + f_4(\phi, X) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta},$$

$$\mathcal{M}_4^{\mu\nu\alpha\beta} = \left[\frac{1}{2} f_{4,X}(\phi, X) + \tilde{f}_4(\phi) \right] \tilde{F}^{\mu\nu} \tilde{F}^{\alpha\beta}.$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \mathcal{E}^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \quad : \text{Scalar kinetic term} \quad \tilde{F} = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad : \text{Parity-odd electromagnetic invariant}$$

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad : \text{Maxwell invariant} \quad Y = \nabla_\mu \phi \nabla^\nu \phi F^{\mu\alpha} F_{\nu\alpha} \quad : \text{Derivative coupling}$$

Setup

- Action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R + \mathcal{L}_{\text{SVT}}^{(2)} + \mathcal{L}_{\text{SVT}}^{(3)} + \mathcal{L}_{\text{SVT}}^{(4)} \right]$$

$$\mathcal{L}_{\text{SVT}}^{(2)} = f_2(\phi, X, F, \tilde{F}, Y), \quad \text{---} \quad f_2 = F \quad (\text{Einstein-Maxwell theory is recovered})$$

$$\mathcal{L}_{\text{SVT}}^{(3)} = \mathcal{M}_3^{\mu\nu} \nabla_\mu \nabla_\nu \phi,$$

$$\mathcal{M}_3^{\mu\nu} = \left[f_3(\phi, X) g_{\rho\sigma} + \tilde{f}_3(\phi, X) \nabla_\rho \phi \nabla_\sigma \phi \right] \tilde{F}^{\mu\rho} \tilde{F}^{\nu\sigma},$$

$$\mathcal{L}_{\text{SVT}}^{(4)} = \mathcal{M}_4^{\mu\nu\alpha\beta} \nabla_\mu \nabla_\alpha \phi \nabla_\nu \nabla_\beta \phi + f_4(\phi, X) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta},$$

$$\mathcal{M}_4^{\mu\nu\alpha\beta} = \left[\frac{1}{2} f_{4,X}(\phi, X) + \tilde{f}_4(\phi) \right] \tilde{F}^{\mu\nu} \tilde{F}^{\alpha\beta}.$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \mathcal{E}^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \quad : \text{Scalar kinetic term} \quad \tilde{F} = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad : \text{Parity-odd electromagnetic invariant}$$

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad : \text{Maxwell invariant} \quad Y = \nabla_\mu \phi \nabla^\nu \phi F^{\mu\alpha} F_{\nu\alpha} \quad : \text{Derivative coupling}$$

Setup

- Action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R + \mathcal{L}_{\text{SVT}}^{(2)} + \mathcal{L}_{\text{SVT}}^{(3)} + \mathcal{L}_{\text{SVT}}^{(4)} \right]$$

$$\mathcal{L}_{\text{SVT}}^{(2)} = f_2(\phi, X, F, \tilde{F}, Y), \quad \text{---} \quad f_2 = F \quad (\text{Einstein-Maxwell theory is recovered})$$

$$\begin{aligned} \mathcal{L}_{\text{SVT}}^{(3)} &= \mathcal{M}_3^{\mu\nu} \nabla_\mu \nabla_\nu \phi, \\ \mathcal{M}_3^{\mu\nu} &= \left[f_3(\phi, X) g_{\rho\sigma} + \tilde{f}_3(\phi, X) \nabla_\rho \phi \nabla_\sigma \phi \right] \tilde{F}^{\mu\rho} \tilde{F}^{\nu\sigma}, \\ \mathcal{L}_{\text{SVT}}^{(4)} &= \mathcal{M}_4^{\mu\nu\alpha\beta} \nabla_\mu \nabla_\alpha \phi \nabla_\nu \nabla_\beta \phi + f_4(\phi, X) L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \\ \mathcal{M}_4^{\mu\nu\alpha\beta} &= \left[\frac{1}{2} f_{4,X}(\phi, X) + \tilde{f}_4(\phi) \right] \tilde{F}^{\mu\nu} \tilde{F}^{\alpha\beta}. \end{aligned}$$

More general interactions

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \mathcal{E}^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \quad : \text{Scalar kinetic term} \quad \tilde{F} = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad : \text{Parity-odd electromagnetic invariant}$$

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad : \text{Maxwell invariant} \quad Y = \nabla_\mu \phi \nabla^\nu \phi F^{\mu\alpha} F_{\nu\alpha} \quad : \text{Derivative coupling}$$

Setup II

- Static and spherically symmetric background

$$ds^2 = \bar{g}_{\mu\nu} x^\mu x^\nu = -f(r)dt^2 + h(r)^{-1}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$\phi = \bar{\phi}(r), \quad \bar{A}_\mu = (A_0(r), 0, 0, -\textcolor{red}{P} \cos \theta) \quad P: \text{Magnetic charge}$$

- Condition for eliminating higher derivatives from the equations of motion

M. Kitagawa, N. Tsukamoto and R. Kase (2026)

$$\tilde{f}_4 = \frac{3}{2} f_{4,X}$$

Overview of Stability Analysis

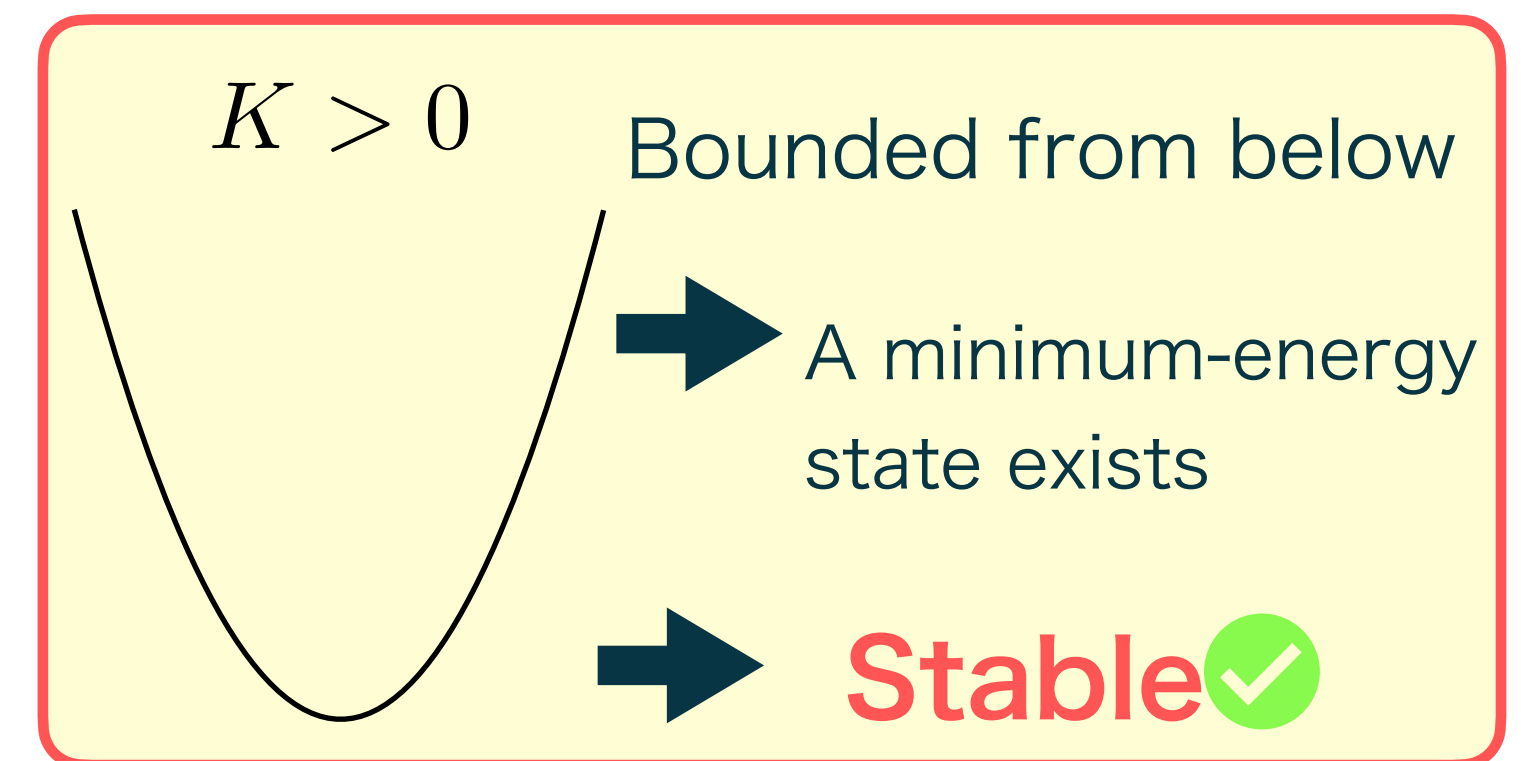
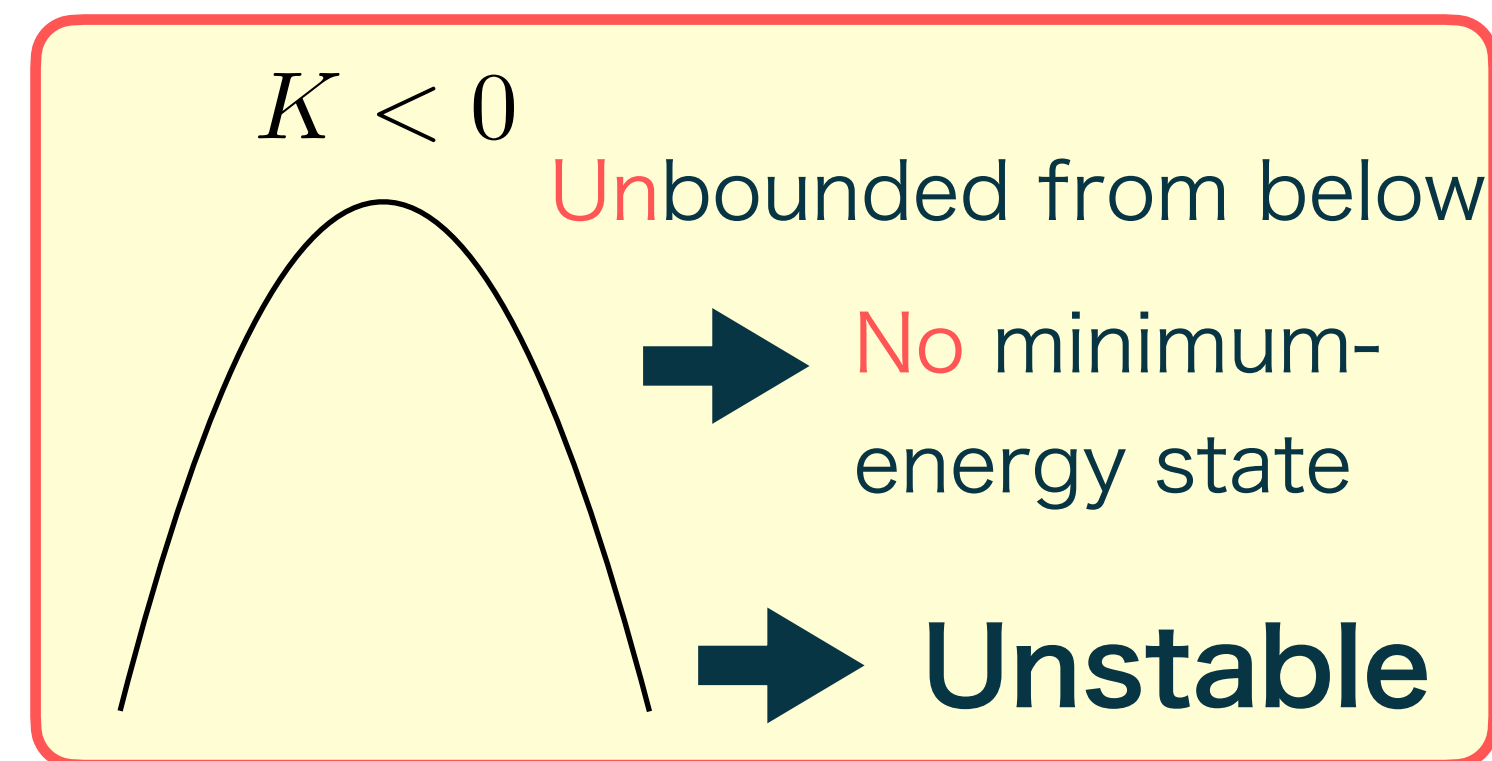
- Introduce perturbations to the fields and study their dynamics.

$$\phi = \bar{\phi} + \delta\phi$$

- **Instabilities**

- **Ghost instability**

$$\mathcal{L}_{\text{ex.}}^{(2)} = K \dot{\delta\phi}^2 + G \delta\phi'^2 \dots$$

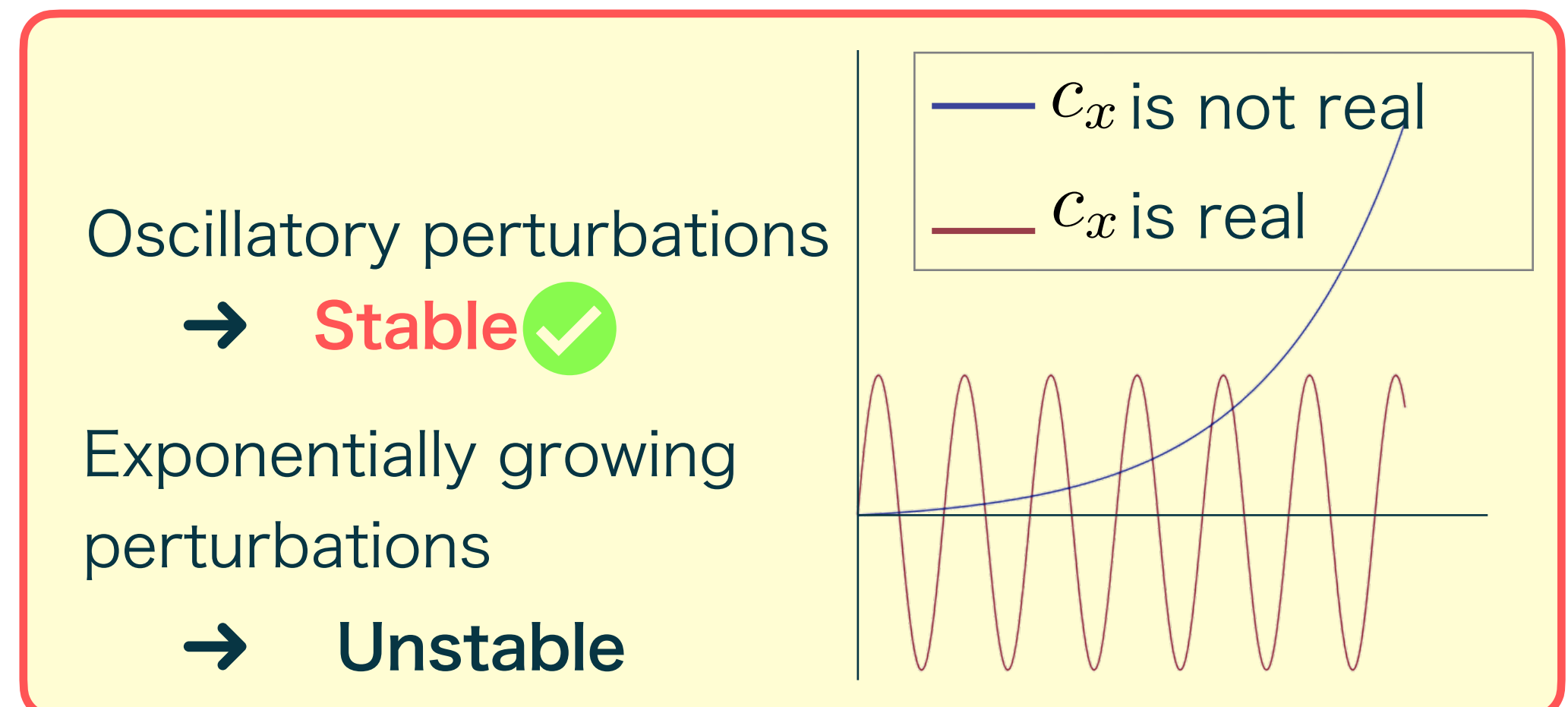


- **Gradient Instability**

Assume a plane-wave solution

$$\delta\phi = \tilde{\delta\phi} \exp[i(\omega t - kx)]$$

For the propagation speed $c_x \equiv \frac{\omega}{k}$
we require c_x to be real.



Overview of Stability Analysis

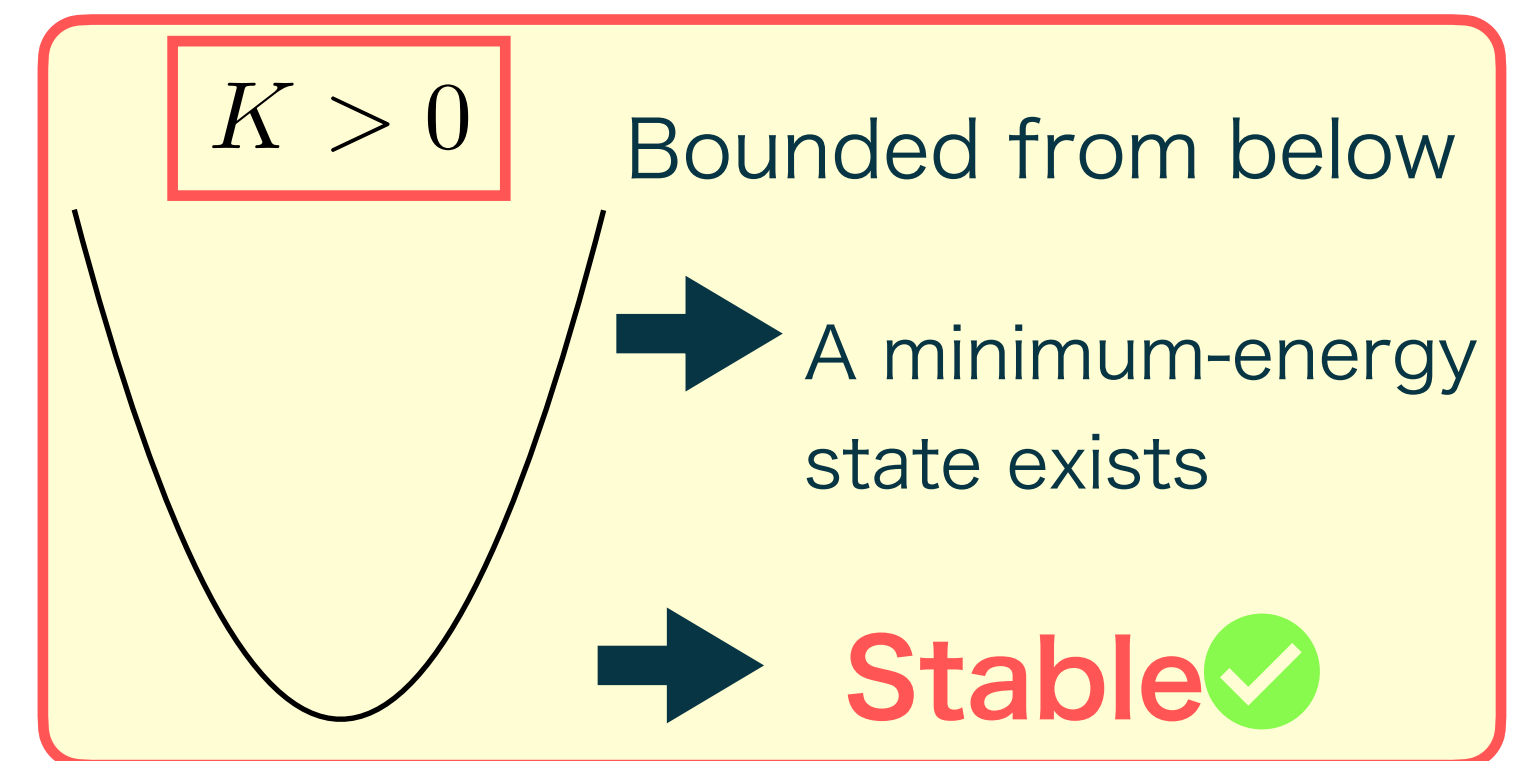
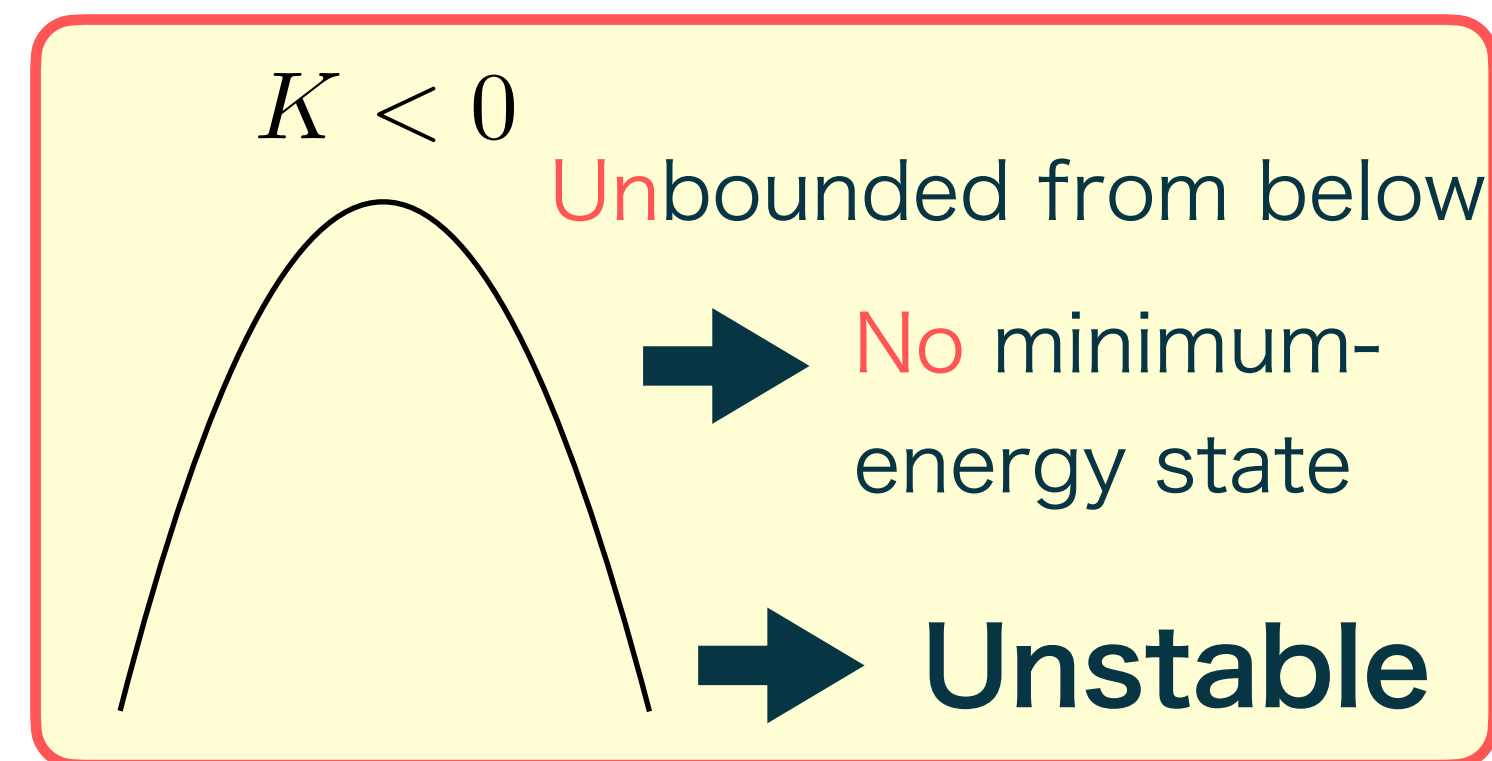
- Introduce perturbations to the fields and study their dynamics.

$$\phi = \bar{\phi} + \delta\phi$$

- **Instabilities**

- **Ghost instability**

$$\mathcal{L}_{\text{ex.}}^{(2)} = K\dot{\delta\phi}^2 + G\delta\phi'^2 \dots$$

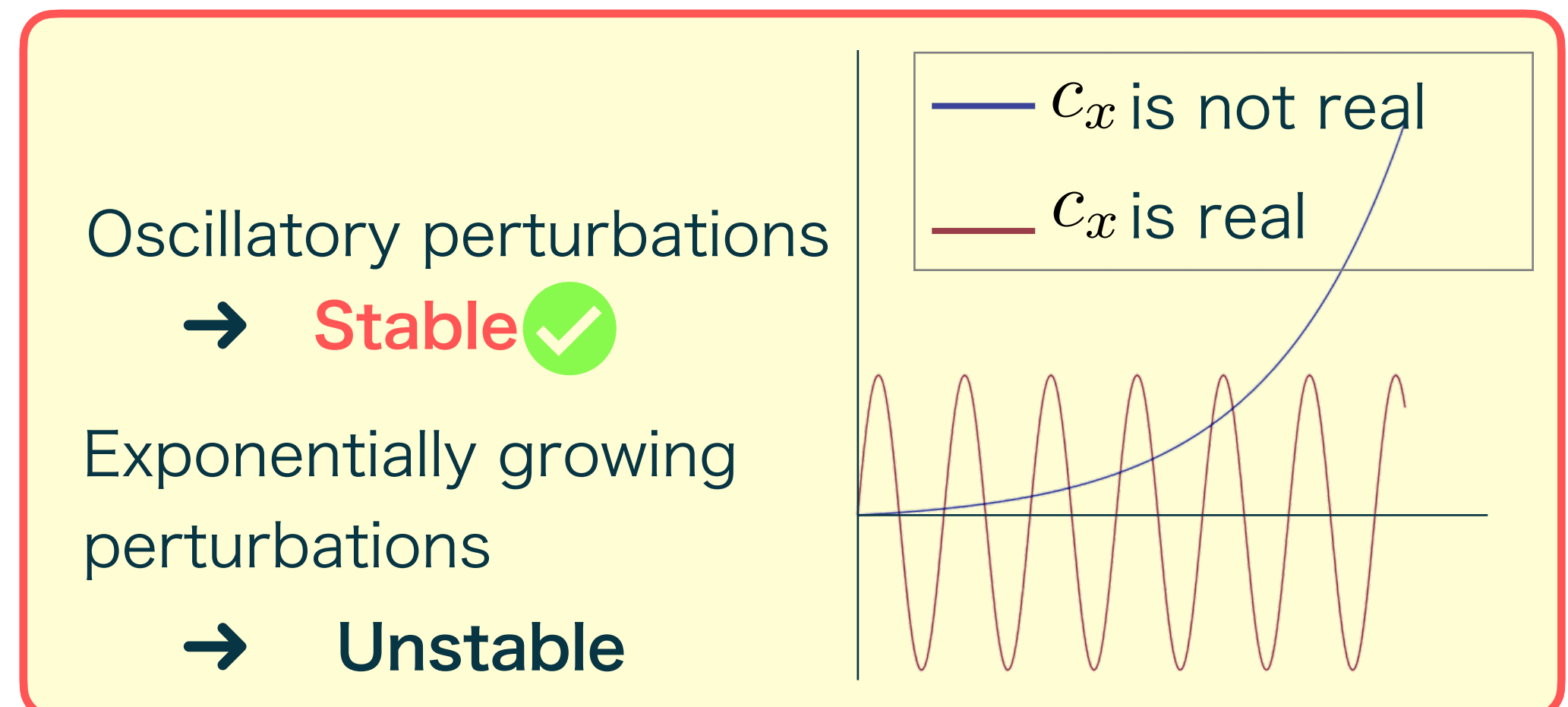


- **Gradient Instability**

Assume a plane-wave solution

$$\delta\phi = \tilde{\delta\phi} \exp[i(\omega t - kx)]$$

For the propagation speed $c_x \equiv \frac{\omega}{k}$
we require c_x to be real.



Overview of Stability Analysis

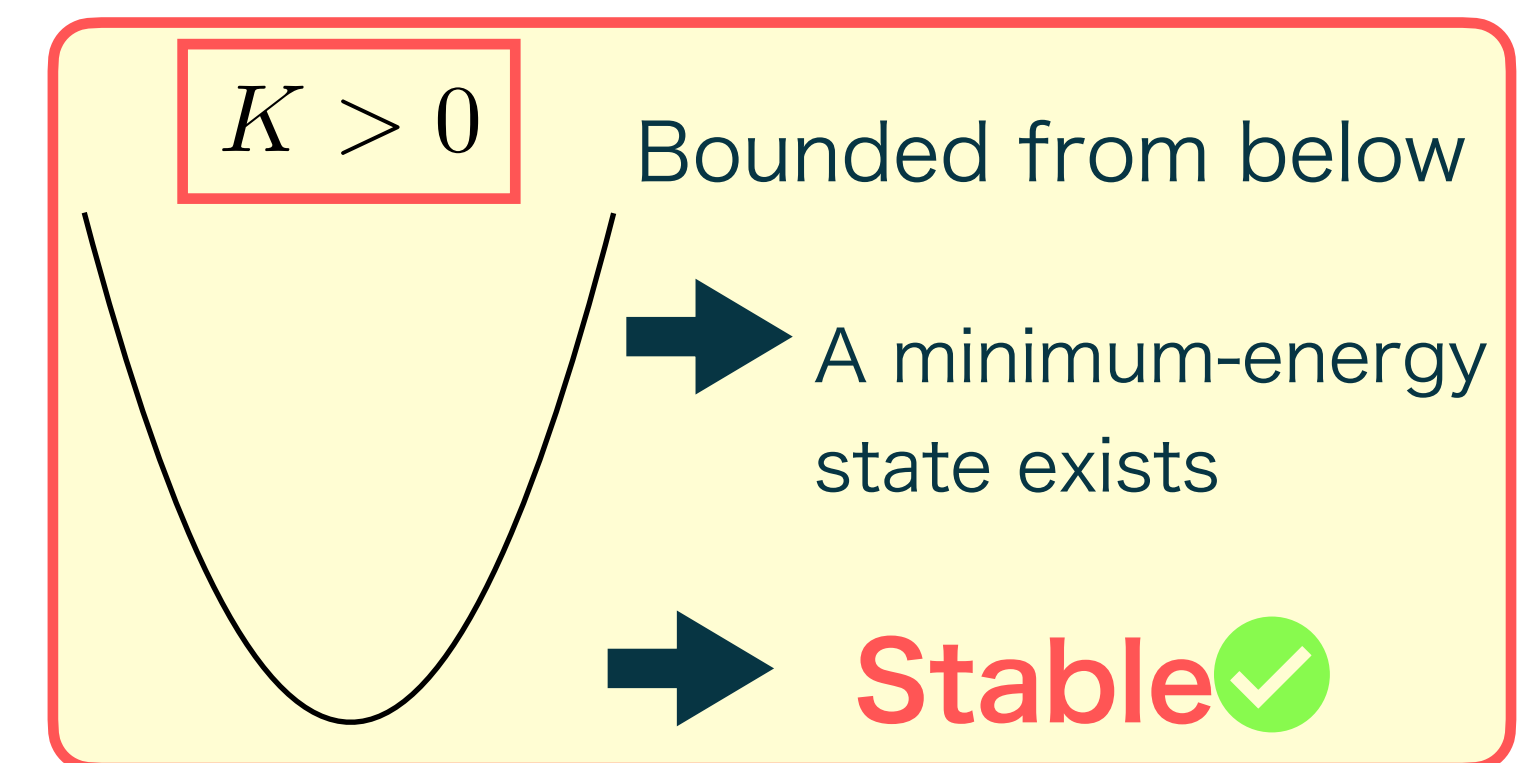
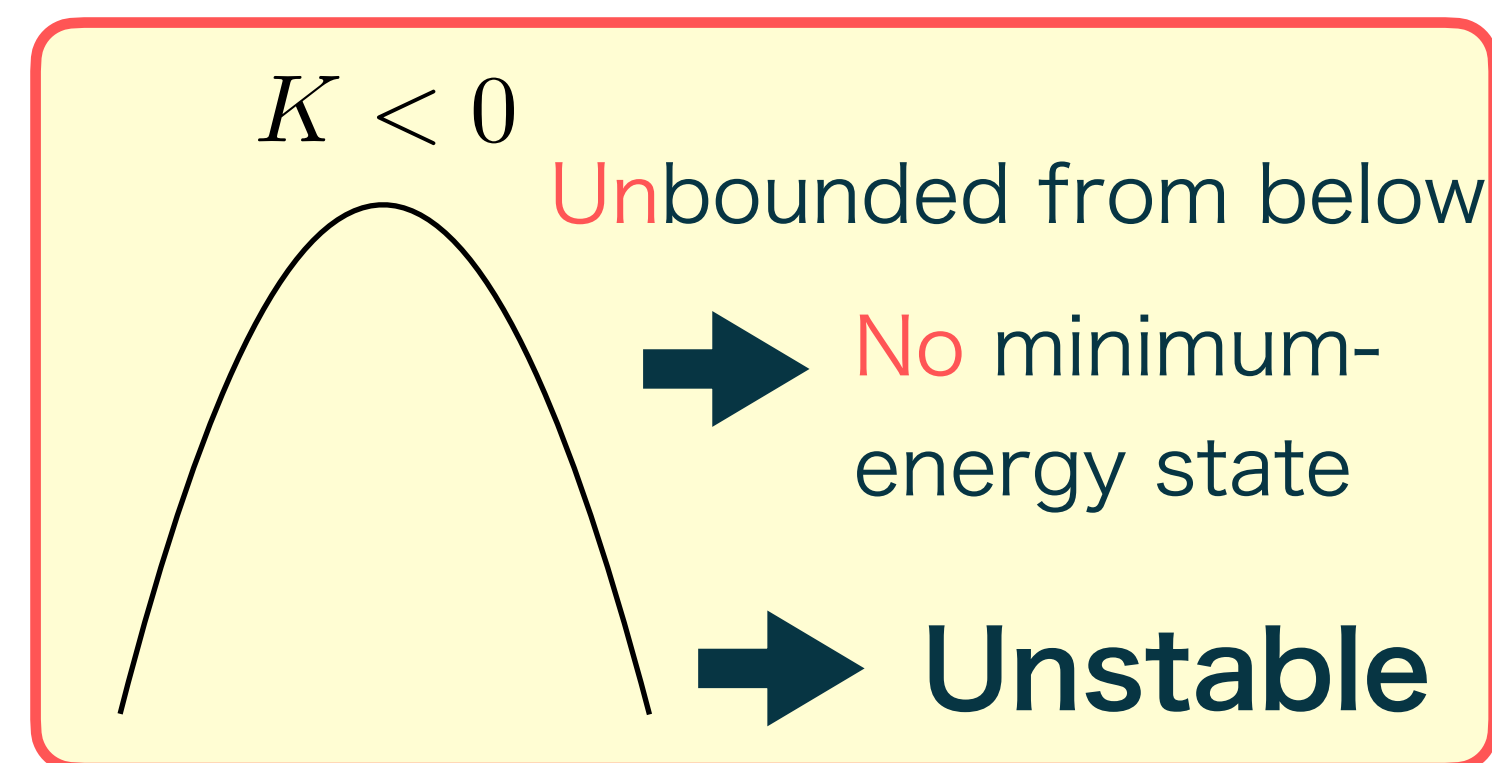
- Introduce perturbations to the fields and study their dynamics.

$$\phi = \bar{\phi} + \delta\phi$$

- **Instabilities**

- **Ghost instability**

$$\mathcal{L}_{\text{ex.}}^{(2)} = K\dot{\delta\phi}^2 + G\delta\phi'^2 \dots$$

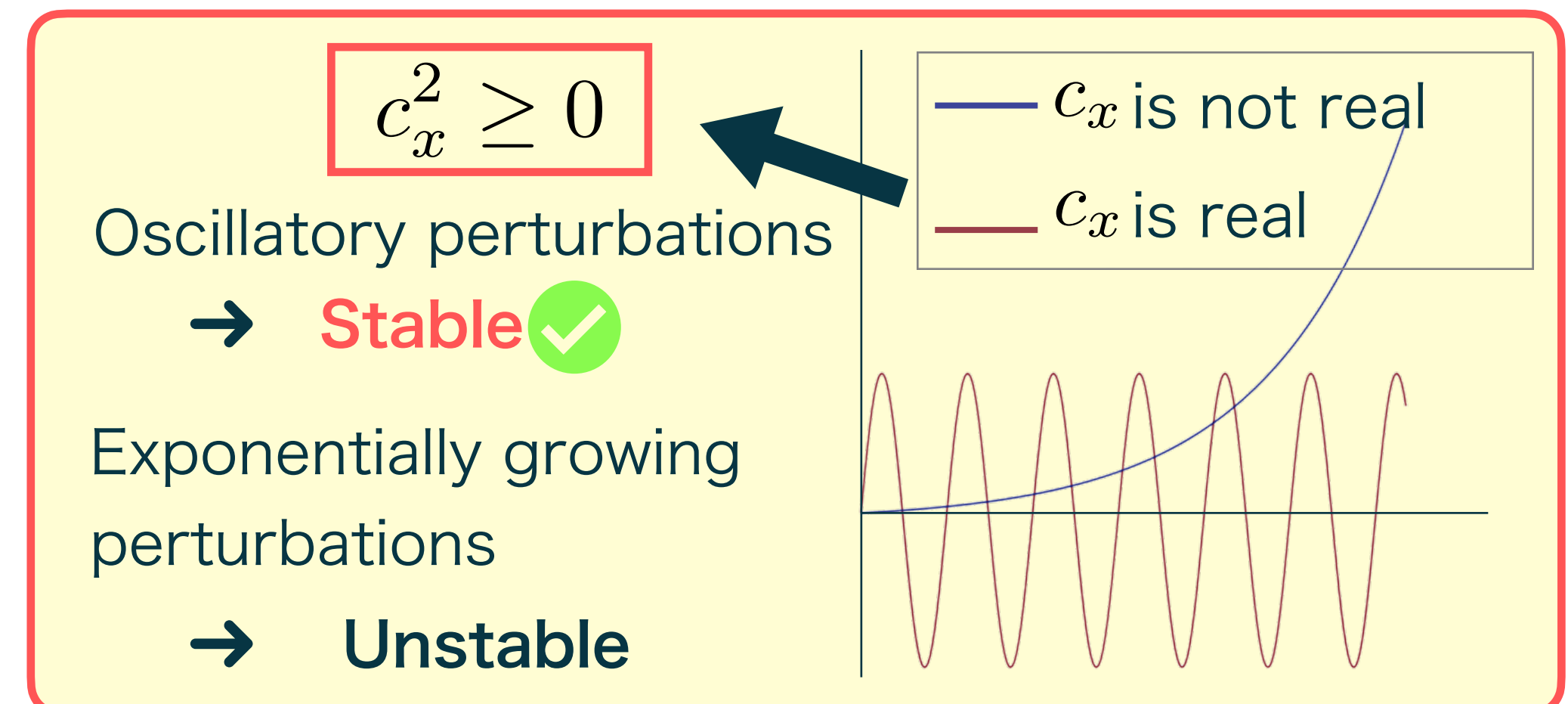


- **Gradient Instability**

Assume a plane-wave solution

$$\delta\phi = \tilde{\delta\phi} \exp[i(\omega t - kx)]$$

For the propagation speed $c_x \equiv \frac{\omega}{k}$ we require c_x to be real.



Contents

1. Introduction

Background, Previous Work, Motivation and Objectives

2. Black Hole Perturbation Analysis

Setup, Overview of Stability Analysis

3. Results and Summary

Results: General Stability Conditions

- The independent stability conditions are:

$$\text{STC}[1] : \alpha_6(r) < 0,$$

$$\text{STC}[2] : a_4(r) \Xi_1(r) < 0,$$

$$\text{STC}[3] : 0 < a_4(r) \Delta_V(r),$$

$$\text{STC}[4] : \frac{a_4(r) S_2(r)}{b_4(r)} < 0,$$

$$\text{STC}[5] : 0 < \alpha_7(r),$$

No-ghost conditions

$$\text{STC}[6] : \det \mathcal{G} < 0,$$

Gradient stability
(radial direction)

$$\text{STC}[7] : 0 < v_1(r),$$

$$\text{STC}[8] : 0 < -\alpha_7(r) \alpha_9(r) + \frac{1}{4} \alpha_8(r)^2,$$

$$\text{STC}[9] : 0 < a_4(r) v_9(r),$$

$$\text{STC}[10] : 0 < -\mathcal{E}(r),$$

$$\text{STC}[11] : \mathcal{S}(r) < 0.$$

Gradient stability
(angular direction)

Results: General Stability Conditions

- The independent stability conditions are:

$$\Xi_1(r) = f(r)m_{16}(r)^2 - 2v_9(r)a_4(r).$$

$$\Delta_V(r) = v_6(r)^2 f(r) - 8a_4(r)v_{10}(r).$$

$$S_2(r) = \Xi_2(r) + \Xi_3(r).$$

$$\begin{aligned} \Xi_2(r) = ra_4(r) \left[r \left(\frac{f'(r)}{f(r)} + \frac{h'(r)}{h(r)} + \frac{b'_4(r)}{b_4(r)} \right) - 2 \right] \\ - 2r^2 a'_4(r) - f(r)m_{16}(r)P \\ - \frac{1}{2}A'_0(r)r^2 v_6(r). \end{aligned}$$

$$\Xi_3(r) = \frac{f(r)^2 h(r) b_4(r) r^2 \alpha_7(r)}{a_4(r)}.$$

$$\begin{aligned} \det \mathcal{G} = & - \frac{h(r)^2 r^6 v_1(r) a_4(r)^2 \alpha_6(r) \Delta_V(r) b_4(r)^2}{4f(r)\phi'(r)^2 \alpha_7(r) \Xi_1(r) Q(r)^2} \\ & \times \left[\frac{1}{4} b_4(r)^2 \left(v_1(r) \alpha_6(r) + \frac{1}{4} v_6(r)^2 \right) f(r)^3 \right. \\ & - \left\{ \left(r \alpha_6(r) - \frac{1}{4} d_2(r) \phi'(r) \right) v_1(r) + \frac{1}{4} r v_6(r)^2 \right. \\ & \quad \left. \left. + \frac{1}{8} v_6(r) \phi'(r) v_4(r) \right\} a_4(r) b_4(r) f(r)^2 \right. \\ & + \left\{ \left[\frac{1}{4} \left(-r d_2(r) + \frac{1}{2} A'_0(r) v_4(r) + c_2(r) \right) \phi'(r) \right. \right. \\ & \quad \left. \left. + r^2 \alpha_6(r) + \frac{1}{8} f'(r) b_4(r) \right] v_1(r) \right. \\ & \quad \left. \left. + \frac{1}{4} \left(r v_6(r) + \frac{1}{2} \phi'(r) v_4(r) \right)^2 \right\} a_4(r)^2 f(r) \right. \\ & \left. \left. - \frac{1}{4} a_4(r)^3 f'(r) v_1(r) r \right] \right]. \end{aligned}$$

$$\begin{aligned} \mathcal{E}(r) = e_4(r) + \frac{a_6(r) d_2(r)}{a_4(r)} \\ - \frac{m_{f5}(r)^2}{4a_4(r)^2 v_9(r)}. \end{aligned}$$

$$\mathcal{S}(r) = \widehat{M}_{33}(r) - \frac{\widehat{M}_{23}(r)^2}{\mathcal{E}(r)}.$$

$$\begin{aligned} \widehat{M}_{23}(r) = w_{KM}(r) M_{12}^{(0)} - u_{KM}(r) M_{23}^{(0)} \\ - \frac{M_{24}^{(0)} \left[w_{KM}(r) M_{14}^{(0)} - u_{KM}(r) M_{34}^{(0)} \right]}{M_{44}^{(0)}}. \end{aligned}$$

$$\begin{aligned} \widehat{M}_{33}(r) = w_{KM}(r)^2 M_{11}^{(1)} - 2u_{KM}(r) w_{KM}(r) M_{13}^{(1)} \\ + u_{KM}(r)^2 M_{33}^{(1)} \\ - \frac{\left[w_{KM}(r) M_{14}^{(0)} - u_{KM}(r) M_{34}^{(0)} \right]^2}{M_{44}^{(0)}}. \end{aligned}$$

$$- \frac{1}{4} \alpha_8(r)^2,$$

STC[1]

STC[2]

STC[3]

STC[4]

STC[5]

No-ghost conditions

Gradient stability
(radial direction)

Gradient stability
(angular direction)

Results: General Stability Conditions

- The independent stability conditions are:

$$\Xi_1(r) = f(r)m_{16}(r)^2 - 2v_9(r)a_4(r).$$

$$\Delta_V(r) = v_6(r)^2 f(r) - 8a_4(r)v_{10}(r).$$

$$S_2(r) = \Xi_2(r) + \Xi_3(r).$$

$$\begin{aligned} \Xi_2(r) = ra_4(r) \left[r \left(\frac{f'(r)}{f(r)} + \frac{h'(r)}{h(r)} + \frac{b'_4(r)}{b_4(r)} \right) - 2 \right] \\ - 2r^2 a'_4(r) - f(r)m_{16}(r)P \\ - \frac{1}{2}A'_0(r)r^2 v_6(r). \end{aligned}$$

$$\Xi_3(r) = \frac{f(r)^2 h(r) b_4(r) r^2 \alpha_7(r)}{a_4(r)}.$$

$$\begin{aligned} \det \mathcal{G} = & - \frac{h(r)^2 r^6 v_1(r) a_4(r)^2 \alpha_6(r) \Delta_V(r) b_4(r)^2}{4f(r)\phi'(r)^2 \alpha_7(r) \Xi_1(r) Q(r)^2} \\ & \times \left[\frac{1}{4} b_4(r)^2 \left(v_1(r) \alpha_6(r) + \frac{1}{4} v_6(r)^2 \right) f(r)^3 \right. \\ & - \left\{ \left(r \alpha_6(r) - \frac{1}{4} d_2(r) \phi'(r) \right) v_1(r) + \frac{1}{4} r v_6(r)^2 \right. \\ & \quad \left. \left. + \frac{1}{8} v_6(r) \phi'(r) v_4(r) \right\} a_4(r) b_4(r) f(r)^2 \right. \\ & + \left\{ \left[\frac{1}{4} \left(-r d_2(r) + \frac{1}{2} A'_0(r) v_4(r) + c_2(r) \right) \phi'(r) \right. \right. \\ & \quad \left. \left. + r^2 \alpha_6(r) + \frac{1}{8} f'(r) b_4(r) \right] v_1(r) \right. \\ & \quad \left. \left. + \frac{1}{4} \left(r v_6(r) + \frac{1}{2} \phi'(r) v_4(r) \right)^2 \right\} a_4(r)^2 f(r) \right. \\ & \left. - \frac{1}{4} a_4(r)^3 f'(r) v_1(r) r \right]. \end{aligned}$$

$$\begin{aligned} \mathcal{E}(r) = e_4(r) + \frac{a_6(r) d_2(r)}{a_4(r)} \\ - \frac{m_{f5}(r)^2}{4a_4(r)^2 v_9(r)}. \end{aligned}$$

$$\mathcal{S}(r) = \widehat{M}_{33}(r) - \frac{\widehat{M}_{23}(r)^2}{\mathcal{E}(r)}.$$

$$\begin{aligned} \widehat{M}_{23}(r) = w_{KM}(r) M_{12}^{(0)} - u_{KM}(r) M_{23}^{(0)} \\ - \frac{M_{24}^{(0)} \left[w_{KM}(r) M_{14}^{(0)} - u_{KM}(r) M_{34}^{(0)} \right]}{M_{44}^{(0)}}. \end{aligned}$$

$$\begin{aligned} \widehat{M}_{33}(r) = w_{KM}(r)^2 M_{11}^{(1)} - 2u_{KM}(r) w_{KM}(r) M_{13}^{(1)} \\ + u_{KM}(r)^2 M_{33}^{(1)} \\ - \frac{\left[w_{KM}(r) M_{14}^{(0)} - u_{KM}(r) M_{34}^{(0)} \right]^2}{M_{44}^{(0)}}. \end{aligned}$$

$$- \frac{1}{4} \alpha_8(r)^2,$$

The explicit expressions are quite lengthy.

No-ghost conditions

Gradient stability
(radial direction)

Gradient stability
(angular direction)

Results: Application to Specific Models

- ▶ We consider the following two models:

$$X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi, \quad F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

$$f_2 = F + X, \quad \tilde{f}_3 = \frac{g_3}{X}, \quad f_3 = f_4 = 0$$

- ▶ Case 1: $g_3 = g_3(\phi)$ Case 2: $g_3 = g_3(\phi, X)$

- ▶ For further simplification, we focus on the near-horizon region.
- ▶ We examine
 - ▶ No-ghost conditions
 - ▶ No-gradient-instability conditions
- ▶ We use the background solution obtained by M. Kitagawa et al. (2026).

Results: Case 1 — $g_3 = g_3(\phi)$

► Independent stability conditions

$$\frac{2}{3} < \mu, \quad 0 < \mu < 1 : \text{a dimensionless parameter characterizing the electric charge}$$

$$M_{\text{Pl}}^2 r_h^8 - 2P^4 g_3(r_h)^2 < 0,$$

$$0 < \frac{1}{4} M_{\text{Pl}}^2 r_h^8 - 3g_3(r_h)^2 \left(\mu - \frac{2}{3} \right) M_{\text{Pl}}^2 P^2 r_h^2 + P^4 g_3(r_h)^2,$$

$$- \frac{1}{2} r_h^8 (\mu - 1) M_{\text{Pl}}^2 + P^4 g_3(r_h)^2 \mu < 0.$$

Results: Case 1 — $g_3 = g_3(\phi)$

► Independent stability conditions

$$\frac{2}{3} < \mu,$$

$0 < \mu < 1$: a dimensionless parameter
characterizing the electric charge

$$M_{\text{Pl}}^2 r_h^8 - 2P^4 g_3(r_h)^2 < 0,$$

$$0 < \frac{1}{4} M_{\text{Pl}}^2 r_h^8 - 3g_3(r_h)^2 \left(\mu - \frac{2}{3} \right) M_{\text{Pl}}^2 P^2 r_h^2 + P^4 g_3(r_h)^2,$$

$$-\frac{1}{2} r_h^8 (\mu - 1) M_{\text{Pl}}^2 + P^4 g_3(r_h)^2 \mu < 0.$$

Results: Case 1 — $g_3 = g_3(\phi)$

► Independent stability conditions

$$\frac{2}{3} < \mu,$$

$0 < \mu < 1$: a dimensionless parameter
characterizing the electric charge

$$M_{\text{Pl}}^2 r_h^8 - 2P^4 g_3(r_h)^2 < 0,$$

$$0 < \frac{1}{4} M_{\text{Pl}}^2 r_h^8 - 3g_3(r_h)^2 \left(\mu - \frac{2}{3} \right) M_{\text{Pl}}^2 P^2 r_h^2 + P^4 g_3(r_h)^2,$$

$$-\frac{1}{2} r_h^8 (\mu - 1) M_{\text{Pl}}^2 + P^4 g_3(r_h)^2 \mu < 0.$$

Cannot be satisfied

Results: Case 1 — $g_3 = g_3(\phi)$

► Independent stability conditions

$$\frac{2}{3} < \mu,$$

$0 < \mu < 1$: a dimensionless parameter
characterizing the electric charge

$$M_{\text{Pl}}^2 r_h^8 - 2P^4 g_3(r_h)^2 < 0,$$

$$0 < \frac{1}{4} M_{\text{Pl}}^2 r_h^8 - 3g_3(r_h)^2 \left(\mu - \frac{2}{3} \right) M_{\text{Pl}}^2 P^2 r_h^2 + P^4 g_3(r_h)^2,$$

$$-\frac{1}{2} r_h^8 (\mu - 1) M_{\text{Pl}}^2 + P^4 g_3(r_h)^2 \mu < 0.$$

Cannot be satisfied

✓ No stable parameter region exists for Case 1 near the horizon.

Results: Case 2 — $g_3 = g_3(\phi, X)$

► Independent stability conditions

$$\frac{2}{3} < \mu, \quad \Delta = [M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2]^2 + 36M_{\text{Pl}}^4 P^4 \left(\mu - \frac{2}{3}\right) g_{3,X}(r_h) r_h^6 g_3(r_h).$$
$$0 < \sqrt{\Delta} - 6P^4 g_3(r_h)^2,$$

$$0 < 3P^2 r_h^4 M_{\text{Pl}}^2 - \frac{72M_{\text{Pl}}^4 P^4 r_h^6 g_3(r_h)^2 \left(\mu - \frac{2}{3}\right)}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}}, \quad 0 < \mu < 1$$

$$1 - \mu + \frac{12P^4 g_3(r_h)^2 d}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}} < 0.$$

Results: Case 2 — $g_3 = g_3(\phi, X)$

► Independent stability conditions

$$\frac{2}{3} < \mu, \quad \Delta = [M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2]^2 + 36M_{\text{Pl}}^4 P^4 \left(\mu - \frac{2}{3}\right) g_{3,X}(r_h) r_h^6 g_3(r_h).$$
$$0 < \sqrt{\Delta} - 6P^4 g_3(r_h)^2,$$

$$0 < 3P^2 r_h^4 M_{\text{Pl}}^2 - \frac{72M_{\text{Pl}}^4 P^4 r_h^6 g_3(r_h)^2 \left(\mu - \frac{2}{3}\right)}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}}, \quad 0 < \mu < 1$$

$$1 - \mu + \frac{12P^4 g_3(r_h)^2 d}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}} < 0.$$

Results: Case 2 — $g_3 = g_3(\phi, X)$

► Independent stability conditions

$$\frac{2}{3} < \mu, \quad \Delta = [M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2]^2 + 36M_{\text{Pl}}^4 P^4 \left(\mu - \frac{2}{3}\right) g_{3,X}(r_h) r_h^6 g_3(r_h).$$
$$0 < \sqrt{\Delta} - 6P^4 g_3(r_h)^2,$$

$$0 < 3P^2 r_h^4 M_{\text{Pl}}^2 - \frac{72M_{\text{Pl}}^4 P^4 r_h^6 g_3(r_h)^2 \left(\mu - \frac{2}{3}\right)}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}}, \quad 0 < \mu < 1$$

$$1 - \mu + \frac{12P^4 g_3(r_h)^2 d}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}} < 0.$$

Cannot be satisfied

Results: Case 2 — $g_3 = g_3(\phi, X)$

► Independent stability conditions

$$\frac{2}{3} < \mu, \quad \Delta = [M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2]^2 + 36M_{\text{Pl}}^4 P^4 \left(\mu - \frac{2}{3}\right) g_{3,X}(r_h) r_h^6 g_3(r_h).$$
$$0 < \sqrt{\Delta} - 6P^4 g_3(r_h)^2,$$

$$0 < 3P^2 r_h^4 M_{\text{Pl}}^2 - \frac{72M_{\text{Pl}}^4 P^4 r_h^6 g_3(r_h)^2 \left(\mu - \frac{2}{3}\right)}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}}, \quad 0 < \mu < 1$$

$$1 - \mu + \frac{12P^4 g_3(r_h)^2 d}{M_{\text{Pl}}^2 r_h^8 + 4P^4 g_3(r_h)^2 + \sqrt{\Delta}} < 0.$$

Cannot be satisfied

✓ No stable parameter region exists for Case 2 near the horizon.

Summary

- Derived general stability conditions for U(1) gauge-invariant SVT theories including magnetic charge.
- For the cases $g_3 = g_3(\phi)$ and $g_3 = g_3(\phi, X)$, we derived the near-horizon stability conditions and found no parameter region satisfying all stability conditions.
- Future work
 - ▶ Apply the general stability conditions to other specific models and investigate the behavior away from the asymptotic regions.
 - ▶ Study black hole quasinormal modes and other gravitational-wave signatures in stable parameter regions, and investigate their distinguishability from GR.

Thank you for your attention.

Appendix A: Black Hole Perturbation Analysis

- Introduce perturbations to each field.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + \delta A_\mu, \quad \phi = \bar{\phi} + \delta\phi.$$

- Decompose the perturbations into odd- and even-parity sectors according to their transformation under parity.

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^{(\text{odd})} + \delta g_{\mu\nu}^{(\text{even})}, \quad \delta A_\mu = \delta A_\mu^{(\text{odd})} + \delta A_\mu^{(\text{even})}, \quad \delta\phi = \delta\phi^{(\text{even})}.$$

- For $P = 0$, the odd- and even-parity sectors decouple:

$$\mathcal{L} = \mathcal{L}_{\text{odd} \times \text{odd}} + \mathcal{L}_{\text{even} \times \text{even}}$$

Appendix A: Black Hole Perturbation Analysis

- Introduce perturbations to each field.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + \delta A_\mu, \quad \phi = \bar{\phi} + \delta\phi.$$

- Decompose the perturbations into odd- and even-parity sectors according to their transformation under parity.

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^{(\text{odd})} + \delta g_{\mu\nu}^{(\text{even})}, \quad \delta A_\mu = \delta A_\mu^{(\text{odd})} + \delta A_\mu^{(\text{even})}, \quad \delta\phi = \delta\phi^{(\text{even})}.$$

- For $P = 0$, the odd- and even-parity sectors decouple:
- For $P \neq 0$, the magnetic background induces couplings between the odd- and even-parity perturbations, giving rise to a mixed-parity sector in the quadratic Lagrangian:

$$\mathcal{L} = \underbrace{\mathcal{L}_{\text{odd}}}_{\text{odd} \times \text{odd}} + \underbrace{\mathcal{L}_{\text{even}}}_{\text{even} \times \text{even}} + \underbrace{\mathcal{L}_{\text{mix}}}_{\text{odd} \times \text{even}}$$

Appendix A: Black Hole Perturbation Analysis

- Introduce perturbations to each field.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + \delta A_\mu, \quad \phi = \bar{\phi} + \delta\phi.$$

- Decompose the perturbations into odd- and even-parity sectors according to their transformation under parity.

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^{(\text{odd})} + \delta g_{\mu\nu}^{(\text{even})}, \quad \delta A_\mu = \delta A_\mu^{(\text{odd})} + \delta A_\mu^{(\text{even})}, \quad \delta\phi = \delta\phi^{(\text{even})}.$$

- For $P = 0$, the odd- and even-parity sectors decouple:
- For $P \neq 0$, the magnetic background induces couplings between the odd- and even-parity perturbations, giving rise to a mixed-parity sector in the quadratic Lagrangian:

$$\mathcal{L} = \underbrace{\mathcal{L}_{\text{odd}}}_{\text{odd} \times \text{odd}} + \underbrace{\mathcal{L}_{\text{even}}}_{\text{even} \times \text{even}} + \underbrace{\mathcal{L}_{\text{mix}}}_{\text{odd} \times \text{even}}$$

➡ The odd- and even-parity perturbations must therefore be treated simultaneously.

Appendix A: Dynamical Degrees of Freedom

	Scalar	Vector	Tensor
Field components	1	4	10



Extract the dynamical degrees of freedom

DOF	1	2	2
symbol	$\delta\phi^{(\text{even})}$	$V^{(\text{even})}, \delta A^{(\text{odd})}$	$\psi^{(\text{even})}, \chi^{(\text{odd})}$

- Action for the dynamical degrees of freedom

$$\mathcal{S} = \sum_l \int dt dr \left(\dot{\vec{\chi}}^t \mathbf{K} \dot{\vec{\chi}} + \vec{\chi}'^t \mathbf{G} \vec{\chi}' + \vec{\chi}'^t \mathbf{S} \vec{\chi} + \vec{\chi}^t \mathbf{M} \vec{\chi} \right)$$

$$\vec{\chi} = \begin{pmatrix} \psi \\ \delta\phi \\ V \\ \chi \\ \delta A \end{pmatrix}$$

✓ The general stability conditions can be derived from these matrices.