

Dynamical Gravitational-Wave Lensing by Gravitational Atoms

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Based on work in preparation (arXiv:26xx.xxxxx)

Gravitational Atom from BH Superradiance

- ▶ Superradiant instability leads to the formation of dense boson clouds around rotating black holes when $\omega < m\Omega_H$; mass fraction up to ~ 0.1
- ▶ Fine-structure constant $\alpha = GM\mu$

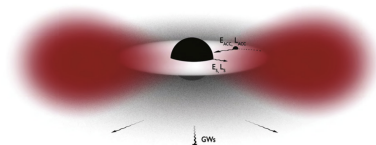
$$\alpha \simeq 0.075 \left(\frac{M_{\text{BH}}}{10M_{\odot}} \right) \left(\frac{\mu}{10^{-12}\text{eV}} \right)$$

Compton wavelength $\sim$ gravitational radius

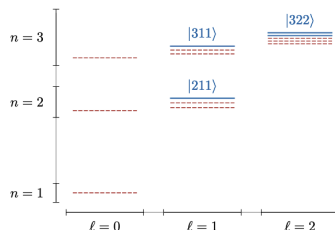
- ▶ Eigenstates and wavefunctions similar to the hydrogen atom

$$\omega_{nlm} = \omega_{nlm}^R + i\omega_{nlm}^I \quad \text{Bohr radius } r_0 \sim (\mu\alpha)^{-1} = GM/\alpha^2$$

$$\omega_{nlm}^R = \mu \left(1 - \frac{\alpha^2}{2n^2} + \mathcal{O}(\alpha^4) \right), \quad \omega_{nlm}^I \sim \mu(m\Omega_H - \omega)\alpha^{4\ell+5}$$



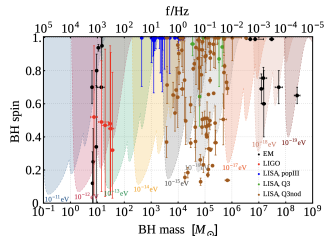
[Brito et al., arXiv:1411.0686]



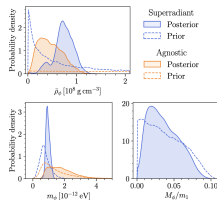
[Baumann et al., arXiv:1908.10370]

Discovering Gravitational Atoms

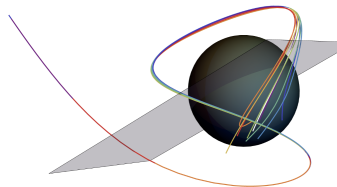
- ▶ BH spin measurements constrain boson masses through superradiant spin-down.
- ▶ Continuous GWs from the cloud probe boson annihilation.
- ▶ Imprints on (EMRI or equal-mass) binary waveforms.
- ▶ EM probes include shadows, photon rings, and axion-induced polarization.
- ▶ Other probes...



BH mass-spin exclusions: Brito et al., arXiv:1706.06311



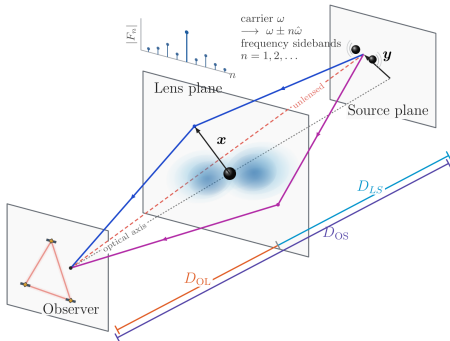
Binary waveform imprint: Roy et al.,
arXiv:2510.17967



Photon-ring probe: Chen et al.,
arXiv:2211.03794

Periodic GW Lensing Signatures From GA

- ▶ The macroscopic cloud sources a gravitational potential that can lens background GWs.
- ▶ Distinctive feature of GA: Coherent interference between occupied levels—here the self-interaction-supported $|211\rangle$ – $|322\rangle$ pair—produces a beat at $\hat{\omega}$ and makes the lens potential periodic.
- ▶ Sideband structure: a passing signal with frequency ω is redistributed into a discrete ladder of *sidebands* at $\omega \pm n\hat{\omega}$; their spacing and amplitudes carry information about the level splitting and relative occupations.



Static GW Lensing Formalism

Setup: GW propagating under the gravitational potential of the lens: $ds^2 = -(1 + 2U)dt^2 + (1 - 2U)dr^2$, with $\mathbf{r} = (\xi, Z)$; the amplification factor is $F(\omega) = \tilde{h}^L / \tilde{h}^U$.

Fermat potential and Time delay

$$\Psi(\xi) = 2 \int U(\xi, Z) dZ, \quad t_d(\xi) = \frac{D_{OS}}{2D_{OL}D_{LS}} |\xi|^2 - \Psi(\xi).$$

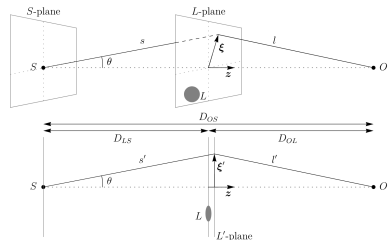
Diffraction integral formula

$$F(\omega) = \frac{\omega}{2\pi i} \frac{D_{OS}}{D_{OL}D_{LS}} \int d^2\xi e^{i\omega t_d(\xi)}.$$

In the geometrical optics limit, $\omega t_d \gg 1$, the diffraction integral is reduced to the sum over images

$$F(\omega) = \sum_j |\mu(\xi_j)|^{1/2} \exp[i\omega t_d(\xi_j) - i\pi n_j],$$

the image positions ξ_j are given by $\partial t_d(\xi) / \partial \xi = 0$.



Dimensionless setup

$$D_{\text{eff}} = \frac{D_{OL}D_{LS}}{D_{OS}}, \quad \mathbf{x} = \frac{\xi - \xi_L}{\xi_*}, \quad \mathbf{y} = -\frac{\xi_L}{\xi_*},$$

$$t_* = \frac{\xi_*^2}{D_{\text{eff}}}, \quad \tau = \frac{t - D_{OL}}{t_*}, \quad w = \omega t_*, \quad \psi = \frac{\Psi}{t_*}.$$

Dimensionless form

$$F(w) = \frac{w}{2\pi i} \int d^2\mathbf{x} \exp \left\{ i w \left[\frac{1}{2} |\mathbf{x} - \mathbf{y}|^2 - \psi(\mathbf{x}) \right] \right\}.$$

Time-Dependent GW Lensing: Projection and Amplification

For a slowly varying lens with $\hat{\omega} \ll \omega$,

Retarded line-of-sight projection

$$\Psi(t - D_{\text{OL}}, \xi) = 2 \int_{-\infty}^{+\infty} U(t - D_{\text{OL}} + Z, \xi, Z) dZ.$$

The retarded time accounts for the finite propagation delay across the lens.

Time-dependent amplification factor [Yang et al., arXiv:2410.16378]

$$F(t, \omega) = \left(1 + \frac{i}{2\omega} \partial_t\right) \left\{ \frac{\omega}{2\pi i D_{\text{eff}}} \int d^2\xi \exp \left[i\omega \left(\frac{|\xi|^2}{2D_{\text{eff}}} - \Psi(t - D_{\text{OL}}, \xi) \right) \right] \right\}.$$

The braces give the instantaneous diffraction integral at a fixed retarded time $t - D_{\text{OL}}$.

The derivative term is typically suppressed by $\hat{\omega}/\omega$.

Dimensionless form

$$F(\tau, w) = \left(1 + \frac{i}{2w} \partial_\tau\right) \left\{ \frac{w}{2\pi i} \int d^2\mathbf{x} e^{iw[\frac{1}{2}|\mathbf{x}-\mathbf{y}|^2 - \psi(\tau, \mathbf{x})]} \right\}.$$

A Multi-Level GA Survives Line-of-Sight Projection

For a single occupied level, the time-dependent metric component oscillates at frequency $\sim 2\mu$. The projected Fermat potential then contains

$$\Psi_{\text{osc}}^{\text{single}} \sim \int (\text{smooth profile on scale } r_0) e^{-2i\mu Z} dZ \sim 0.$$

- ▶ The cloud size is the gravitational Bohr radius: $r_0 = 1/\mu\alpha$.
- ▶ In the weak-coupling regime, the line-of-sight phase varies by $2\mu r_0 \sim \alpha^{-1} \gg 1$.
- ▶ The rapid oscillation is washed out by destructive phase cancellation.

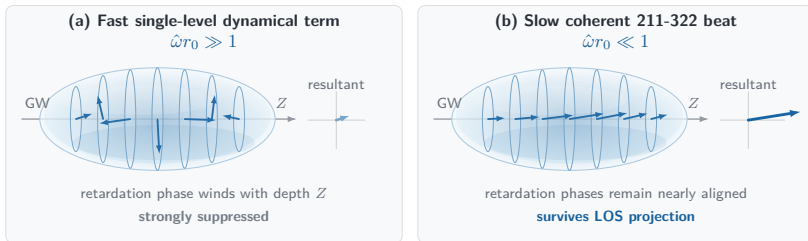
Physical interpretation. Different parts of the oscillating cloud are sampled at different beat phases. The beat-induced lensing potential remains coherent only over a finite range $\sim \hat{\omega}^{-1}$, so their accumulated contributions cancel when $\hat{\omega} r_0 \gg 1$.

Leading unsuppressed dynamical component

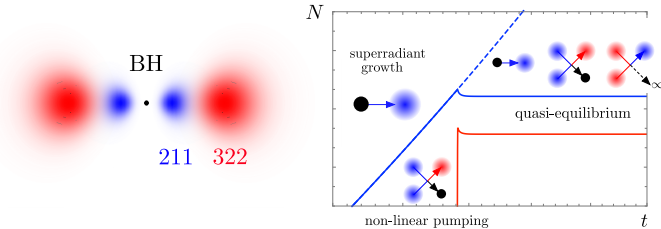
- ▶ A key feature distinguishing GAs from other compact ultralight bosonic objects—such as boson stars or axion halos—is their internal multi-level structure.
- ▶ For Bohr splitting, $\hat{\omega} \simeq \mu\alpha^2$ and $\hat{\omega}r_0 \sim \alpha \ll 1$, so the beat phase remains coherent across the cloud.

$$\Delta\Psi_j \propto U_{\hat{\omega}}(Z_j)e^{-i\hat{\omega}Z_j}, \quad \Psi_{\hat{\omega}} \simeq \sum_j \Delta\Psi_j$$

➡ complex $\Delta\Psi_j$ phasor for each LOS slice j



Self-Interactions Populate a Long-Lived 211–322 Pair



[Baryakhtar et al., arXiv:2011.11646]

Self-interaction seeds the second level

$$V(\phi) \simeq \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4!} \frac{\mu^2}{f_\phi^2} \phi^4 + \dots$$

$$|211\rangle \xrightarrow{\phi^4} |322\rangle.$$

Quasi-equilibrium saturation

$$211 \times 211 \rightarrow 322 \times \text{BH},$$

$$322 \times 322 \rightarrow 211 \times \infty.$$

pumping $\simeq$ depletion

$$\delta \equiv \left(\frac{M_{322}}{M_{211}} \right)^{1/2} \simeq \left[0.04 \left(\frac{\alpha}{0.1} \right)^3 \right]^{1/2}.$$

Moderate coupling (Region B): full $|211\rangle$ saturation + an appreciable $|322\rangle$ occupation.

The Beat Density Is the Leading Dynamical Source

Two coherently occupied hydrogenic levels

$$\Upsilon(t, \mathbf{r}) = \Upsilon_{211}(\mathbf{r})e^{-i(\omega_{211}^R - \mu)t} + \Upsilon_{322}(\mathbf{r})e^{-i(\omega_{322}^R - \mu)t}.$$

The coherent interference between $|211\rangle$ and $|322\rangle$ oscillates at

$$\hat{\omega} \equiv \omega_{322}^R - \omega_{211}^R \simeq \frac{5}{72}\mu\alpha^2.$$

Density decomposition

$$\rho(t, \mathbf{r}) = \rho_{211}(\mathbf{r}) + \rho_{322}(\mathbf{r}) + \rho_{\text{osc}}(t, \mathbf{r}).$$

Leading oscillatory density

$$\rho_{\text{osc}}(t, \mathbf{r}) = \frac{M_{211}\delta}{648\pi r_0^3} e^{-5r/(6r_0)} \left(\frac{r}{r_0}\right)^3 \sin^3 \theta \cos(\hat{\omega}t - \varphi).$$

Dominant stationary density

$$\rho_{211}(\mathbf{r}) = \frac{M_{211}}{64\pi r_0^3} e^{-r/r_0} \left(\frac{r}{r_0}\right)^2 \sin^2 \theta.$$

Fermat Potential of the GA Lens

The Newtonian and projected lensing potentials of the GA lens are decomposed as

$$U(t, \mathbf{r}) = U_{\text{BH}}(\mathbf{r}) + U_0(\mathbf{r}) + U_{\text{osc}}(t, \mathbf{r}),$$

and

$$\psi(t, \mathbf{x}) = \psi_{\text{BH}}(\mathbf{x}) + \psi_0(\mathbf{x}) + \psi_{\text{osc}}(t, \mathbf{x}).$$

Central BH

$$\psi_{\text{BH}}(\mathbf{x}) = \frac{R_E^2}{\xi_*^2} \ln |\mathbf{x}|, \quad R_E = \sqrt{4GM_{\text{BH}}D_{\text{eff}}}.$$

Static cloud

The dominant $|211\rangle$ density gives

$$\psi_0(\mathbf{x}) = \frac{\alpha \bar{M}_{211}}{16\pi\mu t_* \bar{r}_0^5} \int d^2\mathbf{x}' \ln |\mathbf{x} - \mathbf{x}'| \mathcal{S}_0(\mathbf{x}').$$

In the far field,

$$\psi_0(\mathbf{x}) \simeq \bar{M}_{211} \psi_{\text{BH}}(\mathbf{x}), \quad |\mathbf{x}| \gg \bar{r}_0.$$

From Retardation to a Screened Beat Potential

Periodic beat phasor

$$\psi_{\text{osc}}(t, \mathbf{x}) = \text{Re}[\psi_{\text{osc}}^c(\mathbf{x})e^{-i\hat{\omega}t}].$$

Retarded projection

$$k_z = \kappa \equiv \hat{\omega}\xi_* = 5/72\alpha \implies (\nabla_{\mathbf{x}}^2 - \kappa^2)\psi_{\text{osc}}^c \propto \tilde{\rho}_{\text{osc}}(\mathbf{x}, \kappa).$$

Oscillatory projected potential

$$\psi_{\text{osc}}^c(\mathbf{x}) = -\frac{\alpha \overline{M}_{211} \delta}{162\pi\mu t_* \bar{r}_0^6} \int d^2\mathbf{x}' K_0(\kappa|\mathbf{x} - \mathbf{x}'|) \mathcal{S}_{\text{osc}}(\mathbf{x}', \kappa).$$

Static projected potential

$$G_{\text{stat}}(\mathbf{x} - \mathbf{x}') \propto \ln |\mathbf{x} - \mathbf{x}'|.$$

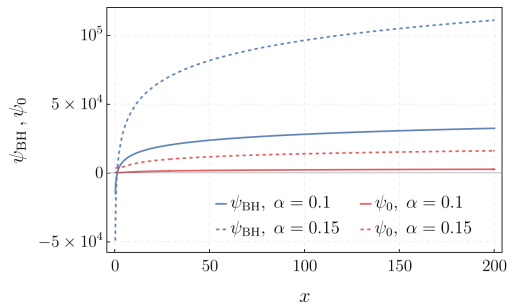
Long-range field; no finite screening scale.

Dynamical beat potential

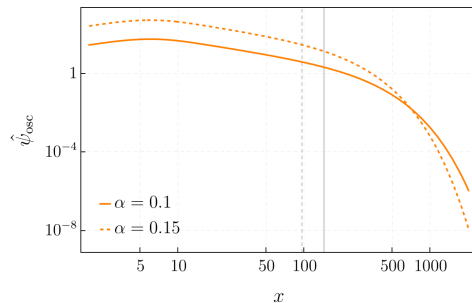
$$G_{\text{dyn}} \propto K_0(\kappa r) \sim \frac{e^{-\kappa r}}{\sqrt{\kappa r}}.$$

Screened beyond the coherence range κ^{-1} .

The Beat Potential Is Screened at Large Radius



Static components: ψ_{BH} and ψ_0 retain the long-range logarithmic behavior.



Beat component: $\hat{\psi}_{\text{osc}}$ decays exponentially beyond the marked screening scale κ^{-1} .

Physical dependence

The 211–322 source contains screened angular multipoles up to $m = 3$. Increasing α changes the beat amplitude and shortens $\kappa^{-1} \propto \alpha^{-1}$.

$$\iota = 0, \mu = 10^{-20} \text{ eV}, D_{\text{eff}} = 1 \text{ kpc}; \alpha = 0.10, 0.15, \text{ equilibrium } \delta.$$

Discrete Sideband Expansion

For a monochromatic periodic lens,

$$\psi(\tau, \mathbf{x}) = \psi_{\text{stat}}(\mathbf{x}) + \hat{\psi}_{\text{osc}}(\mathbf{x}) \cos[\hat{w}\tau - \Theta(\mathbf{x})].$$

Jacobi–Anger expansion

$$\begin{aligned} e^{-i w \hat{\psi}_{\text{osc}}(\mathbf{x}) \cos[\hat{w}\tau - \Theta(\mathbf{x})]} \\ = \sum_{n=-\infty}^{+\infty} \mathcal{M}_n(\mathbf{x}, w) e^{i n \hat{w} \tau}, \end{aligned}$$

with

$$\mathcal{M}_n(\mathbf{x}, w) = i^n J_n[-w \hat{\psi}_{\text{osc}}(\mathbf{x})] e^{-i n \Theta(\mathbf{x})}.$$

Discrete sideband response

$$F(\tau, w) = \sum_{n=-\infty}^{+\infty} F_n(w) e^{i n \hat{w} \tau}.$$

Each coefficient F_n gives the complex amplitude of one frequency-shifted component.

Physical meaning

A periodic lens acts as a phase modulator, mapping one incident component into $\omega - n\hat{w}$, with $n \in \mathbb{Z}$.

Sideband Coefficients

Frequency-resolved amplification

$$F_n(w) = \frac{w}{2\pi i} \int d^2\mathbf{x} \exp \left[\frac{iw}{2} |\mathbf{x} - \mathbf{y}|^2 - iw\psi_{\text{stat}}(\mathbf{x}) \right] \times \left(1 - \frac{n\hat{w}}{2w} \right) \mathcal{M}_n(\mathbf{x}, w).$$

- ▶ When $\hat{\psi}_{\text{osc}} = 0$, only $F_0 = F_{\text{stat}}$ remains.
- ▶ With a dynamical lens, the carrier is modified by $\mathcal{M}_0 = J_0[-w\hat{\psi}_{\text{osc}}] \neq 1$.
- ▶ Nonzero $F_{n \neq 0}$ are the direct sideband signatures of the beat potential.

Static lens

- ▶ Fixed Fermat potential.
- ▶ Diffraction/fringes from a time-independent mass distribution.
- ▶ Frequency response: one carrier component.

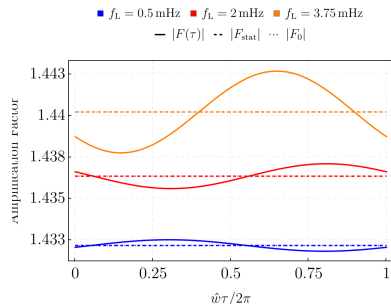
Periodic lens

- ▶ Oscillating component exists (sourced by coherent $|211\rangle$ - $|322\rangle$ beat).
- ▶ The lens transfers power from the carrier to sidebands.
- ▶ Frequency response: a sideband ladder.

Frequency Dependence of the Dynamic Modulation Strength

Dynamical modulation depth

$$\eta(\mathbf{x}) = w\hat{\psi}_{\text{osc}}(\mathbf{x}) \propto f_L, \quad n_{\text{max}} \sim \max_{\mathbf{x}} |\eta(\mathbf{x})|.$$



$$\iota = \pi/4, \quad \mu = 10^{-18} \text{ eV}, \quad \alpha = 0.1, \quad M_{\text{BH}} = 1.34 \times 10^7 M_{\odot},$$

$$D_{\text{eff}} = 0.01 \text{ kpc}, \quad y = \bar{R}_E;$$

$$f_L = 0.5, 2, 3.75 \text{ mHz, corresponding to}$$

$$8\pi GM_{\text{BH}} f_L = 0.825, 3.3, 6.19.$$

- Direct integration shows that the oscillation amplitude grows systematically with frequency.
- All three curves remain in the perturbative regime $\eta \ll 1$. The response is therefore carrier dominated.
- In the following, we focus on high-frequency regime (geometric-optics regime) where the dynamical lensing signatures are more pronounced.

Geometric-optics indicator

$$8\pi GM_{\text{BH}} f_L \simeq 16.5 \left(\frac{\alpha}{0.01} \right) \left(\frac{10^{-20} \text{ eV}}{\mu} \right) \left(\frac{f_L}{1 \text{ mHz}} \right).$$

Geometric-Optics Sideband Formula

In geometric-optics regime where $8\pi GM_{\text{BH}} f_L \gg 1$, the lens-plane integral localizes around image positions $\mathbf{x}_j$ determined by the static delay

$$T_0(\mathbf{x}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}|^2 - \psi_{\text{BH}}(\mathbf{x}) - \psi_0(\mathbf{x}).$$

Each image experiences a local dynamic phase modulation

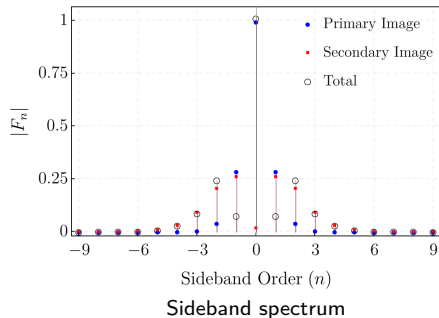
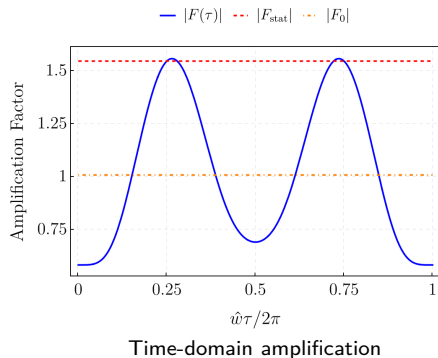
$$F_j(\tau, w) \simeq F_{j,\text{static}}(w) e^{-i w \hat{\psi}_{\text{osc},j} \cos(\hat{w}\tau - \Theta_j)}.$$

Image-level sidebands

$$F_{j,n}(w) = \underbrace{|\mu_j|^{1/2} e^{i w T_0(\mathbf{x}_j) - i \pi n_j}}_{F_{j,\text{static}}(w)} \mathcal{M}_n(\mathbf{x}_j, w).$$

- ▶ Each image carries the usual static GO amplitude and phase, multiplied by a dynamic modulation factor.
- ▶ The full signal is the coherent sum over images and sidebands,
 $F = \sum_{j,n} F_{j,n}.$

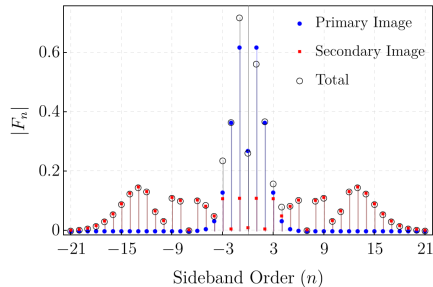
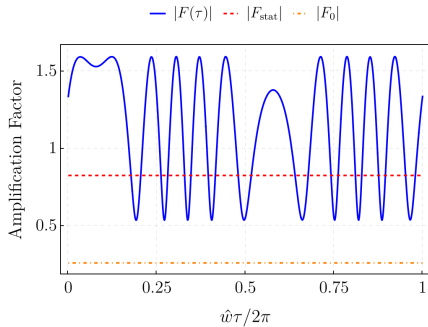
GO Benchmark I: Low-Order Sidebands



GO1: $\iota = 0$, $\mu = 2.00 \times 10^{-20}$ eV, $\alpha = 0.1$, $M_{\text{BH}} = 6.68 \times 10^8 M_{\odot}$, $f_L = 0.03$ Hz, $D_{\text{eff}} = 1$ kpc, $y = \bar{R}_E$, equilibrium δ

- ▶ Sideband support extends to $|n| \sim \mathcal{O}(1)$.
- ▶ The carrier remains dominant, with a modest number of appreciable sidebands.
- ▶ Two-image interference gives a gently non-sinusoidal time-domain profile.

GO Benchmark II: Strong Sideband Excitation



GO2: $\nu = \pi/2$, $\mu = 10^{-20}$ eV, $\alpha = 0.15$, $M_{\text{BH}} = 2.00 \times 10^9 M_{\odot}$, $f_L = 0.05$ Hz, $D_{\text{eff}} = 1$ kpc, $y = \bar{R}_E$,
equilibrium δ

- ▶ Significant support reaches $|n| \sim 20$.
- ▶ The signal develops dense, rapid oscillations on top of the beat envelope.
- ▶ Power is redistributed across a broad sideband comb rather than remaining near the carrier.

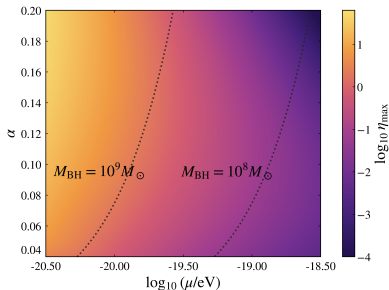
Parameter-Space Maps of Dynamical Effects

Same-host geometry: an SMBH with an ultralight-boson cloud lenses a nearby nuclear GW source,

$$D_{\text{LS}} \sim \text{sub-kpc to kpc} \ll D_{\text{OL}} \simeq D_{\text{OS}}, \text{ so } D_{\text{eff}} \simeq D_{\text{LS}}.$$

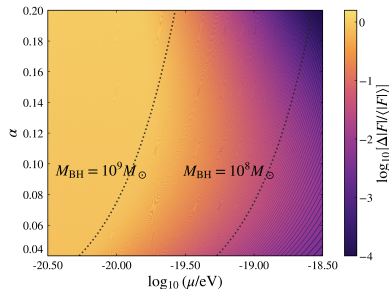
Local sideband strength

$$\eta_{\text{max}} \equiv \max_j \left| w \hat{\psi}_{\text{osc}}(\mathbf{x}_j) \right|.$$



Observable peak-to-peak variation

$$\Delta|F|/\langle|F|\rangle \equiv (|F|_{\text{max}} - |F|_{\text{min}})/\langle|F|\rangle.$$

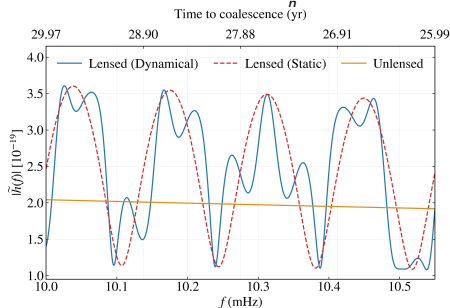


(a) Smaller $\mu \Rightarrow M_{\text{BH}} \uparrow$: deeper GO. (b) Larger $\alpha \Rightarrow \hat{\psi}_{\text{osc}} \uparrow$ but $\kappa^{-1} \downarrow$: peak response at intermediate α .

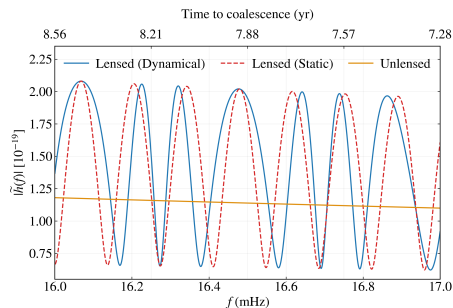
$$D_{\text{eff}} = 0.5 \text{ kpc}, f_{\text{L}} = 0.01 \text{ Hz}, \gamma = \bar{R}_E, \iota = 0, \delta = \delta_{\text{eq}}(\alpha); \text{ dotted curves show constant } M_{\text{BH}}.$$

Dynamically Lensed Waveforms

$$\tilde{h}_{\text{dyn}}^{\text{L}}(f) = \sum_n F_n^{\text{d}}(f + n\hat{f}) \tilde{h}^{\text{U}}(f + n\hat{f}), \quad \tilde{h}_{\text{stat}}^{\text{L}}(f) = F_{\text{stat}}^{\text{d}}(f) \tilde{h}^{\text{U}}(f).$$



Dynamic-ripple window: 10.0–10.55 mHz



Smooth-deformation window: 16.0–17.0 mHz

Source: equal-mass BBH, redshifted $\mathcal{M}_c = 20M_{\odot}$, $D = 1 \text{ Gpc}$, $z_L \simeq z_{\text{BBH}} \simeq 0.20$.

Lens: $\mu = 1.26 \times 10^{-19} \text{ eV}$, $\alpha = 0.18$, $\delta \simeq 0.48$, $D_{\text{eff}} = 0.025 \text{ kpc}$, $y = 0.73\bar{R}_E$, $\iota = 0$.

- ▶ We use a Newtonian binary inspiral, $\tilde{h}^{\text{U}}(f) = A_0 f^{-7/6} e^{i\Phi_{\text{N}}(f)}$.
- ▶ The static lens produces frequency-domain fringes from fixed two-image interference.
- ▶ The dynamical lens either adds fine ripples or smoothly shifts/deforms the static fringe pattern.

Dynamical–Static Mismatch

The key diagnostic is not simply dynamical versus unlensed, but dynamical versus the corresponding static lens.

Faithfulness

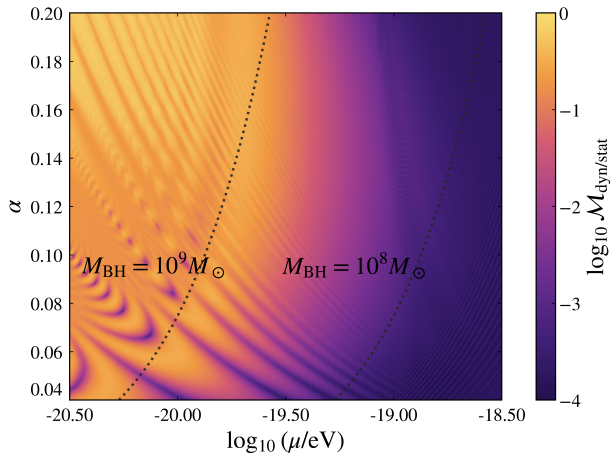
$$\mathcal{F} \equiv \max_{t_c, \phi_c} \frac{(h_{\text{dyn}}^L | h_{\text{stat}}^L)}{\sqrt{(h_{\text{dyn}}^L | h_{\text{dyn}}^L) (h_{\text{stat}}^L | h_{\text{stat}}^L)}}.$$

Mismatch

$$\mathcal{M}_{\text{dyn/stat}} = 1 - \mathcal{F}.$$

- ▶ Maximization over coalescence time and phase removes trivial global shifts.
- ▶ A large mismatch means sidebands cannot be absorbed into a static template.

Mismatch across Parameter Space



Benchmark

- ▶ Lens: $D_{\text{eff}} = 0.2 \text{ kpc}$, $y = \bar{R}_E$, $\iota = 0$, and $\delta = \delta_{\text{eq}}(\alpha)$.
- ▶ Source and band: $\mathcal{M}_c = 20 M_{\odot}$, $\eta = 0.25$, $f \in [0.010, 0.01055] \text{ Hz}$ ($T_{\text{obs}} \simeq 4.0 \text{ yr}$).

Large mismatches occur across broad regions

$$\mathcal{M}_{\text{dyn/stat}} \sim \mathcal{O}(0.1) - \mathcal{O}(1).$$

The sideband imprint is not well absorbed by the corresponding static-lens template.

Main Conclusions

A long-lived, self-interaction-supported 211–322 pair produces a slow beat that survives line-of-sight projection, making periodic GW lensing a probe of the multi-level structure of gravitational atoms—a hallmark feature distinguishing them from other compact ultralight-boson objects.

1. **Coherent projected source.** For the 211–322 pair, $\hat{\omega} \sim \mu\alpha^2$ and $\hat{\omega}r_0 \ll 1$, whereas 2μ oscillations are washed out across the cloud.
2. **Discrete frequency redistribution.** The periodic lens redistributes each incident component at ω among a ladder $\omega \pm n\hat{\omega}$ with $n \in \mathbb{N}$; the spacing tracks the cloud's beat frequency, while the relative sideband weights depend on the level mixture and lens geometry.
3. **Discrimination from static lensing.** Broad parameter regions yield $\mathcal{M}_{\text{dyn/stat}} \sim \mathcal{O}(0.1)\text{--}\mathcal{O}(1)$; beat-induced dynamical signature can be distinguished from a purely static-lensing waveform.

Thank you!

Back up

The 211–322 System Approaches Quasi-Equilibrium

Two-level evolution in Region B

For $\bar{M}_i \equiv M_i/M_{\text{BH}}$, $i = 211, 322$,

$$\dot{\bar{M}}_{211} = \Gamma_{211} \bar{M}_{211} - 2\gamma_{\text{BH}} \bar{M}_{211}^2 \bar{M}_{322} + \gamma_{\infty} \bar{M}_{211} \bar{M}_{322}^2,$$

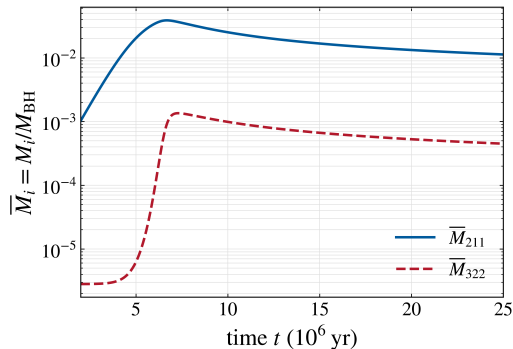
$$\dot{\bar{M}}_{322} = \Gamma_{322} \bar{M}_{322} + \gamma_{\text{BH}} \bar{M}_{211}^2 \bar{M}_{322} - 2\gamma_{\infty} \bar{M}_{211} \bar{M}_{322}^2,$$

$$\dot{\alpha} = -\frac{1}{\alpha} \left(\Gamma_{211} \bar{M}_{211} + 2\Gamma_{322} \bar{M}_{322} \right).$$

Equilibrium mixing

$$\left(\frac{M_{322}}{M_{211}} \right)_{\text{eq}} \simeq \frac{1}{2} \frac{\gamma_{\text{BH}}}{\gamma_{\infty}} \propto \alpha^3,$$

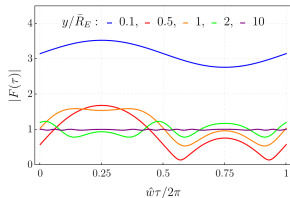
$$\delta \equiv \sqrt{\frac{M_{322}}{M_{211}}} \simeq \left[0.04 \left(\frac{\alpha}{0.1} \right)^3 \right]^{1/2}.$$



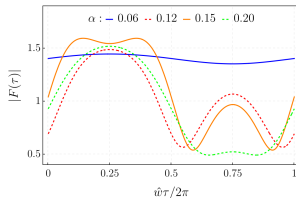
$$\alpha = 0.1, \mu = 10^{-19} \text{ eV}, \tilde{a}_0 = 0.9, \\ M_{\text{BH}} = 1.34 \times 10^8 M_{\odot}, \\ f_{\phi} = 6.97 \times 10^{16} \text{ GeV}.$$

- ▶ The evolution approaches $\delta \simeq 0.2$.
- ▶ The secular drift occurs on a characteristic scale $\sim 10^6$ yr.

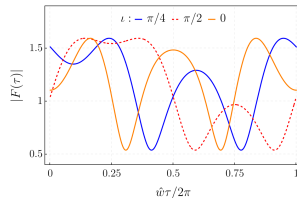
Dependence on Impact Parameter, α , and Inclination



Impact-parameter scan



α scan



Inclination scan

Common parameters: $\mu = 2.00 \times 10^{-20}$ eV, $f_L = 0.04$ Hz, $D_{\text{eff}} = 1$ kpc;

$\mathbf{y} = (y, 0)$ and equilibrium $\delta = \delta_{\text{eq}}(\alpha)$.

- **Panel (a), $y/\bar{R}_E$ scan** ($\alpha = 0.15$, $\iota = \pi/2$): At large impact parameter, screening and secondary-image demagnification make the variability richer but ultimately weaker.
- **Panel (b), α scan** ($y = \bar{R}_E$, $\iota = \pi/2$): A stronger beat competes with a shorter screening length, so the response peaks at intermediate α .
- **Panel (c), ι scan** ($y = \bar{R}_E$, $\alpha = 0.15$): Inclination-induced multipole mixing breaks the near mirror symmetry of the face-on beat cycle.

Reading the Parameter-Space Maps

- ▶ The local sideband strength is characterized by

$$\eta_{\max} = \max_j \left[w \hat{\psi}_{\text{osc}}(\mathbf{x}_j) \right].$$

- ▶ The observable time-domain variation is

$$\frac{\Delta|F|}{\langle|F|\rangle} = \frac{|F|_{\max} - |F|_{\min}}{\langle|F|\rangle}.$$

- ▶ Smaller μ increases the lens mass and pushes the system deeper into geometric optics.
- ▶ Along α , the strongest response lies in an intermediate band: the modulation is strong, but the beat potential is not yet exponentially screened.
- ▶ The time-domain map is finer and less smooth because it probes the coherent sum $|F_+ + F_-|$: small shifts in (μ, α) change the differential image modulation through the image positions and the static two-image phase.

Generic Expression of Dynamical Lensed Waveform

The lensed time-domain strain is

$$h_{\text{dyn}}^{\text{L}}(t) = \int_{-\infty}^{\infty} df \tilde{h}^{\text{U}}(f) e^{-i2\pi ft} F_{\text{d}}(t, f).$$

Using

$$F_{\text{d}}(t, f) = \sum_n F_n^{\text{d}}(f) e^{i2\pi n \hat{f} t},$$

Sideband mixing form of the lensed waveform

$$\tilde{h}_{\text{dyn}}^{\text{L}}(f) = \sum_n F_n^{\text{d}}(f + n\hat{f}) \tilde{h}^{\text{U}}(f + n\hat{f}).$$

In comparison, the static waveform is $\tilde{h}_{\text{stat}}^{\text{L}}(f) = F_{\text{stat}}^{\text{d}}(f) \tilde{h}^{\text{U}}(f)$.

Here t , f and $\hat{f}$ are detector-frame variables.

$$\omega(f) = 2\pi(1 + z_{\text{L}})f, \quad w(f) = \omega(f)t_*, \quad F_{\text{d}}(t, f) \equiv F\left(\frac{t}{1 + z_{\text{L}}}, \omega(f)\right), \quad \hat{f} = \frac{\hat{\omega}}{2\pi(1 + z_{\text{L}})}.$$

The Beat Adds an Image-Dependent Phase to the Chirp

We use a Newtonian binary inspiral, $\tilde{h}^U(f) = A_0 f^{-7/6} e^{i\Phi_N(f)}$ with $\Phi_N(f) = 2\pi f t_c - \Phi_c - \pi/4 + \frac{3}{4}(8\pi G M_c f)^{-5/3}$. For $\hat{f} \ll f$, expand the chirp phase along its time-frequency map:

$$\Phi(f + n\hat{f}) \simeq \Phi(f) + 2\pi n\hat{f} t(f), \quad t(f) = \frac{1}{2\pi} \frac{d\Phi}{df}.$$

Approximate dynamical waveform

$$\tilde{h}_{\text{dyn}}^L(f) \simeq \tilde{h}^U(f) \sum_n F_n^d(f) e^{i2\pi n\hat{f} t(f)}.$$

Geometric-optics interpretation

$$\begin{aligned} \tilde{h}_{\text{dyn}}^L(f) &\simeq \tilde{h}^U(f) \sum_j F_{j,\text{static}}^d(f) e^{i\varphi_j(f)}, \\ e^{i\varphi_j(f)} &= \exp \left[-i\omega \hat{\psi}_{\text{osc},j} \cos \left(2\pi \hat{f} t(f) - \Theta_j \right) \right]. \end{aligned}$$

The observable correction is governed primarily by $\Delta\varphi(f) = \varphi_1(f) - \varphi_2(f)$.

Reading the Waveforms

The GA beat adds an image-dependent phase; only the phase difference between the two images deforms the static fringes:

$$\tilde{h}_{\text{dyn}}^{\text{L}}(f) \simeq \tilde{h}^{\text{U}}(f) \sum_j F_{j,\text{static}}^{\text{d}}(f) e^{i\varphi_j(f)}.$$

$$\varphi_j(f) = -\eta_j(f) \cos[2\pi \hat{f} t(f) - \Theta_j], \quad \Delta\varphi(f) \equiv \varphi_+(f) - \varphi_-(f).$$

Transition parameter: relative-phase change across one static fringe

$$\Xi(f) \equiv \left| \frac{d\Delta\varphi}{df} \right| \Delta f_{\text{stat}}, \quad \Delta f_{\text{stat}} \simeq \frac{1}{|\Delta t_{+-}^{\text{obs}}|}.$$

Ripple regime: $\Xi \gtrsim 1$

$\Delta\varphi$ changes by $\mathcal{O}(1)$ within one fringe, producing visible intrafringe ripples.

With $t(f) \equiv (2\pi)^{-1} d\Phi_{\text{N}}/df$, the rapid phase sweep is

$$\Xi(f) \propto \frac{dt}{df} \propto f^{-11/3}.$$

Lower f : larger Ξ and ripples $\implies$ higher f : smaller Ξ and smooth deformation.

Smooth regime: $\Xi \ll 1$

$\Delta\varphi$ is nearly constant across a fringe, so it mainly shifts or gently deforms the static pattern.