

Probing Hairy Black Holes through Eikonal Ringdown and Null-Geodesic diagnostic

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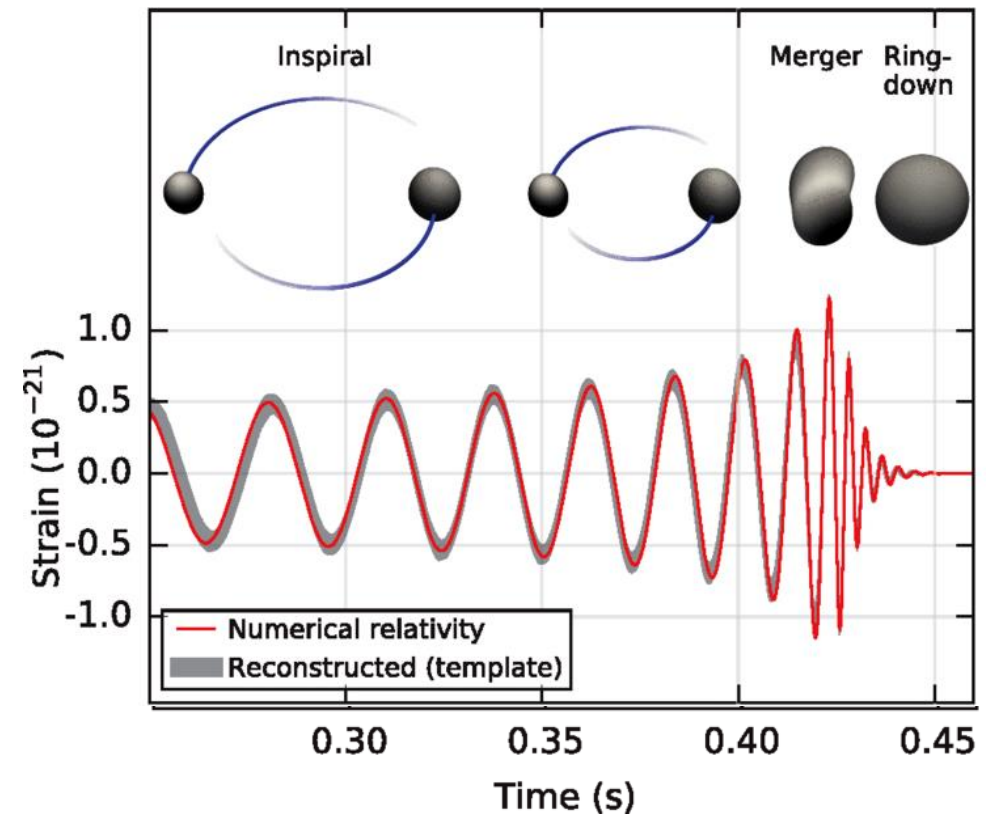
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Background: Black holes & QNM

- In a BH merger, the post-merger remnant relaxes via **ringdown**, whose waveform is well described by a superposition of **quasi-normal modes (QNMs)**.

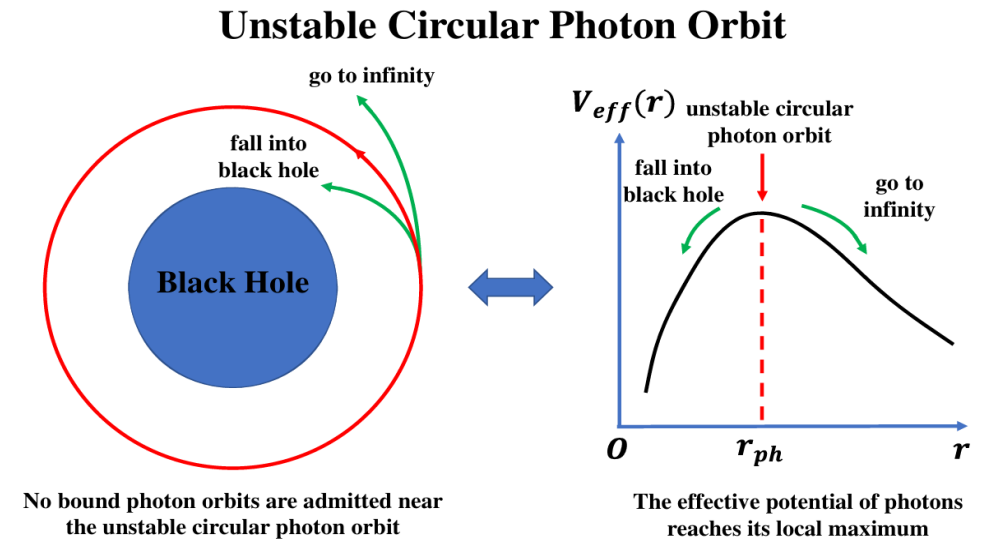


Gravitational wave signal from a binary black hole merger

Adapted from: Yichen Tian, *J. Phys.: Conf. Ser.* **2083**, 022043 (2021).

Background: Black holes & UCOP

- The unstable circular **null** geodesic where photons can orbit is characterized by a **critical impact parameter D** and an **instability rate** given by the **Lyapunov exponent λ** , which controls how fast nearby photon trajectories diverge.

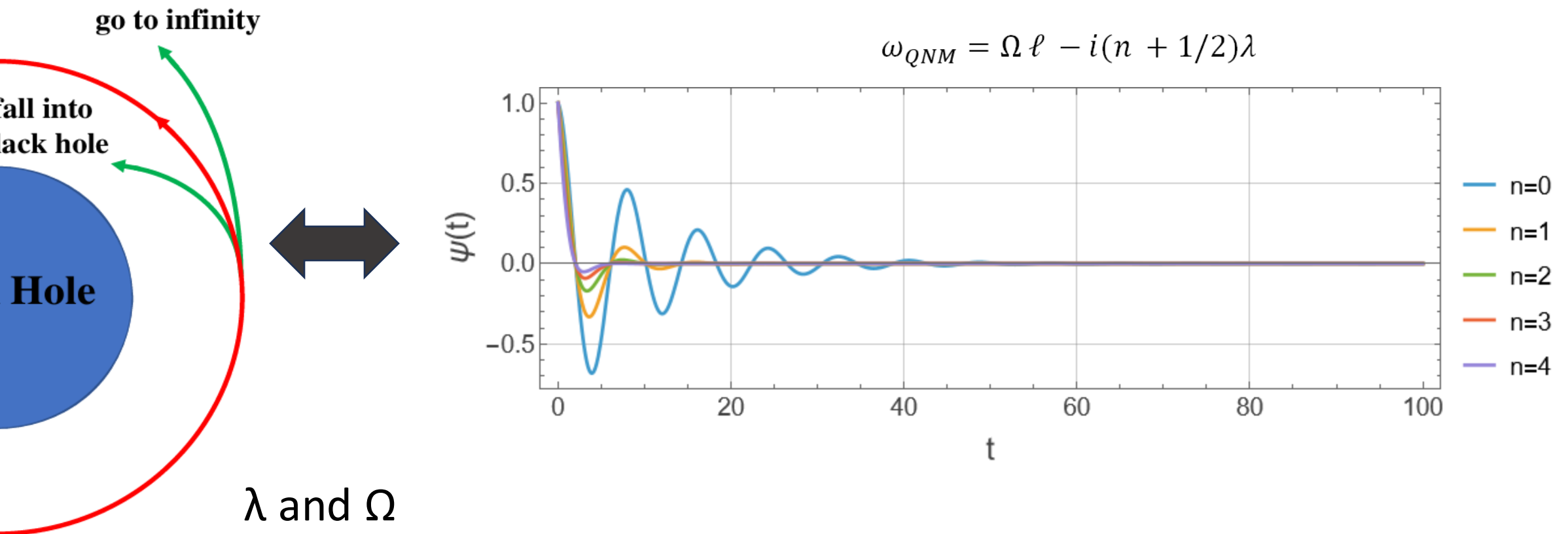


Schematic plot of the unstable circular photon orbits near a black hole

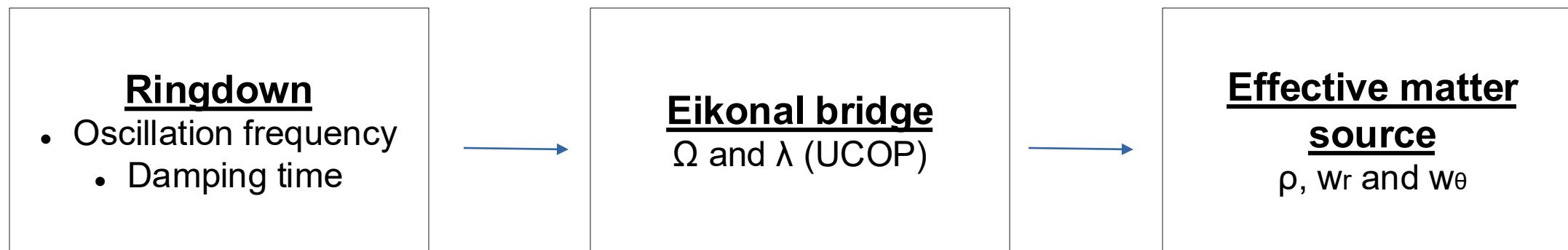
Adapted from: Chen-Kai Qiao and Ming Li, Geometric approach to circular photon orbits and black hole shadows, Phys. Rev. D 106, L021501 (2022).

Background: UCOP and QNM

- Photon ring controls observables (**Eikonal limit**).



What do frequency and damping tell us about the source?



One source-level framework for a broad class of perturbative hairy black holes, without deriving a separate wave equation for every model.

From matter parameters to a ringdown shift

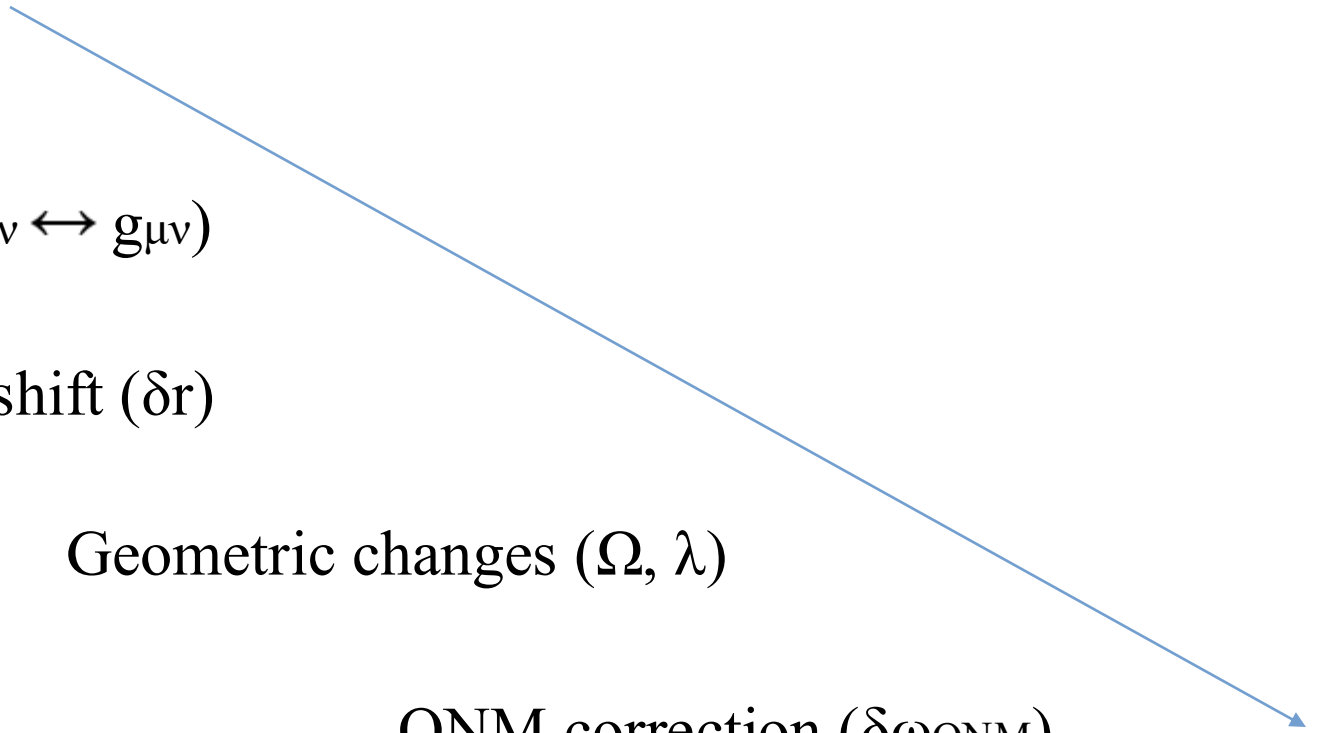
Metric perturbation (ρ, w_r, w_θ)

Einstein equations ($T_{\mu\nu} \leftrightarrow g_{\mu\nu}$)

UCOP shift (δr)

Geometric changes (Ω, λ)

QNM correction ($\delta\omega_{\text{QNM}}$)



Static Spherically Symmetric Setup

General static metric

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2$$

Condition for circular orbits

$$V(r) = 0$$

$$V'(r) = 0$$

QNM frequency

$$\omega_{QNM} = \Omega \ell - i(n + 1/2)\lambda$$

where

$$\Omega := \frac{\dot{\varphi}}{\dot{t}} \stackrel{\circ}{=} \frac{\sqrt{f}}{r}$$

Orbital frequency

$$\lambda^2 := -\frac{V''}{\dot{t}^2} \stackrel{\circ}{=} \frac{f}{2h} \left(\frac{2}{r^2} - \frac{f''}{f} \right)$$

Lyapunov exponent

Spacetime metric construction

1. Perturbation scheme

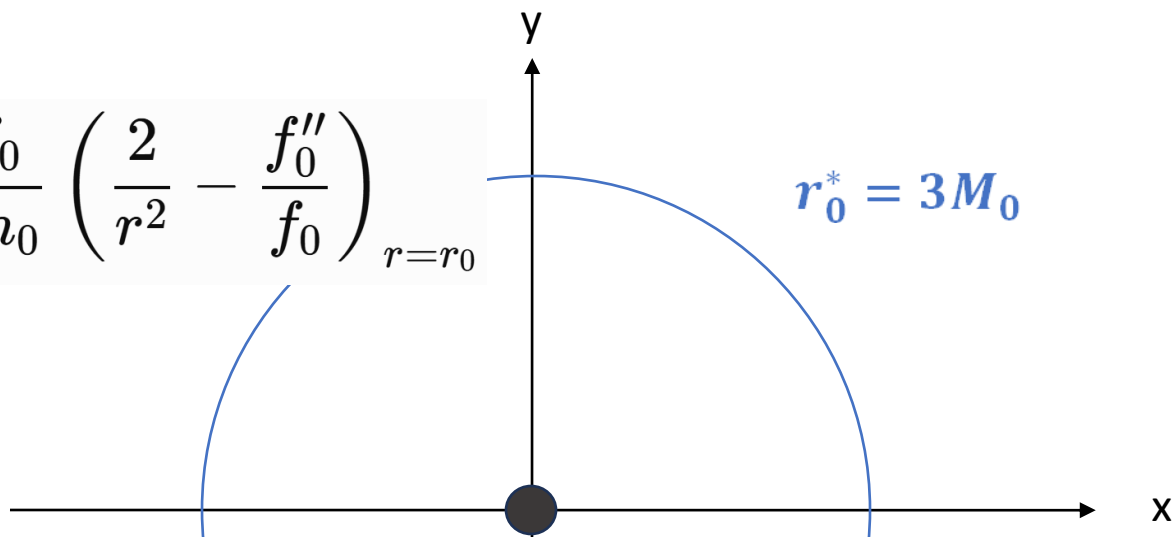
The working scheme is:

0. Schwarzschild solution

$$ds^2 = - \overbrace{\left(1 - \frac{2M_0}{r}\right)}^{f_0(r)} dt^2 + \overbrace{\left(1 - \frac{2M_0}{r}\right)^{-1}}^{h_0(r) = f_0(r)^{-1}} dr^2 + r^2 d\Omega^2$$

$$\Omega_0 := \frac{\dot{\varphi}}{\dot{t}} \stackrel{\circ}{=} \frac{\sqrt{f}}{r}$$

$$\lambda_0^2 := -\frac{V''}{\dot{t}^2} \stackrel{\circ}{=} \frac{f_0}{2h_0} \left(\frac{2}{r^2} - \frac{f_0''}{f_0} \right)_{r=r_0}$$



Spacetime metric construction

1. Perturbation scheme

The working scheme is:

1. Hairy solution

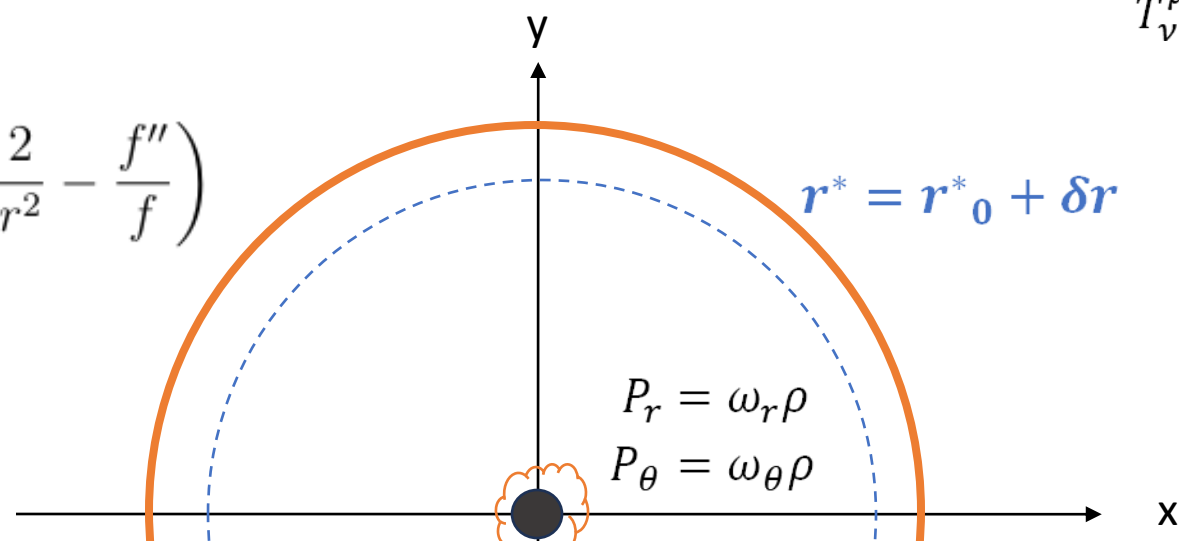
$$\Omega = \frac{\dot{\phi}}{\dot{t}} = \sqrt{\frac{f(r)}{r^2}}$$

$$\lambda^2 := -\frac{V''}{\dot{t}^2} \doteq \frac{f}{2h} \left(\frac{2}{r^2} - \frac{f''}{f} \right)$$

$$ds^2 = -\overbrace{[f_0(r) + \delta f(r)]}^{f(r)} dt^2 + \overbrace{[f_0(r)^{-1} + \delta h(r)]}^{h(r)} dr^2 + r^2 d\Omega^2$$

where $|\rho| \ll 1$

$$T_v^\mu = \text{diag}(-\rho, P_r, P_\theta, P_\theta)$$



QNM Shift & Energy Conditions

Relative shifts

$$\frac{\delta\Omega}{\Omega_0} = \frac{3}{2}\delta f(3M_0)$$

$$\frac{\delta\lambda}{\lambda_0} = \frac{3}{2}\delta f(3M_0) - 4\pi r_0^2 \rho(1 + \omega_\theta)|_{r_0}$$

Where $f(r) = 1 - \frac{2m(r)}{r}$ and $h(r) = f(r)^{-1}$ ($w_r = -1$)

$$m(r) = M_0 + \delta m(r)$$

$$\delta m(r) = -4\pi \int_r^\infty s^2 \rho(s) ds$$

$$\delta f(r) = \frac{8\pi}{r} \int_r^\infty s^2 \rho(s) ds$$

$$\frac{\delta\Omega}{\Omega_0} = \frac{12\pi}{r_0} \int_{r_0}^\infty s^2 \rho(s) ds$$

$$\frac{\delta\lambda}{\lambda_0} = \frac{12\pi}{r_0} \int_{r_0}^\infty s^2 \rho(s) ds - 4\pi r_0^2 \rho(r_0)(1 + w_\theta)$$

QNM Shift & Energy Conditions

The static difference measures $\rho(1 + w_\theta)$

$$\frac{\delta\lambda}{\lambda_0} - \frac{\delta\Omega}{\Omega_0} \simeq -4\pi r_0^2 (\rho + P_\theta) r_0$$

IF $\rho > 0$

the sign is controlled by $1 + w_\theta$

TANGENTIAL NEC

$$\rho + P_\theta = \rho(1 + w_\theta) \geq 0$$

Static ringdown isolates the tangential, not the radial, local source combination.

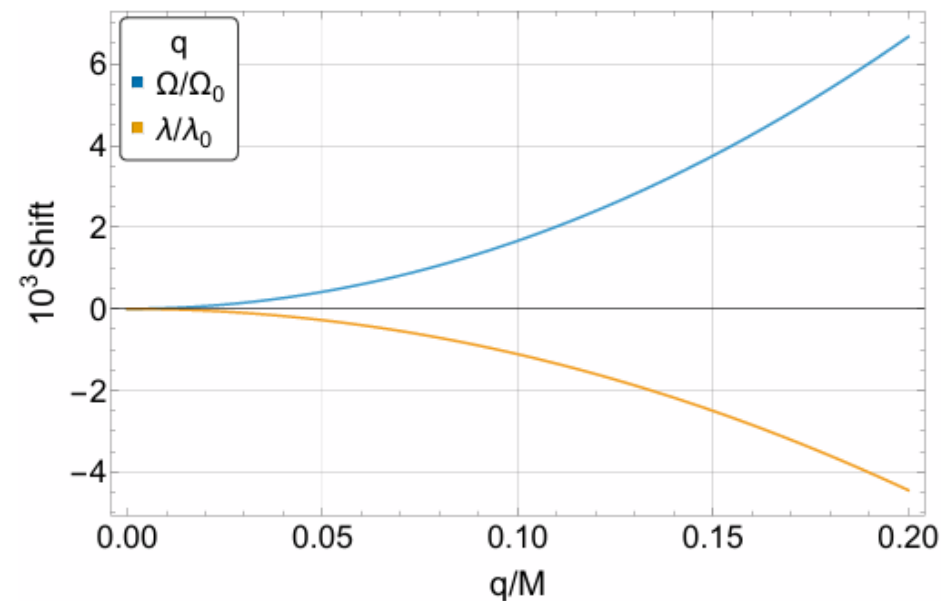
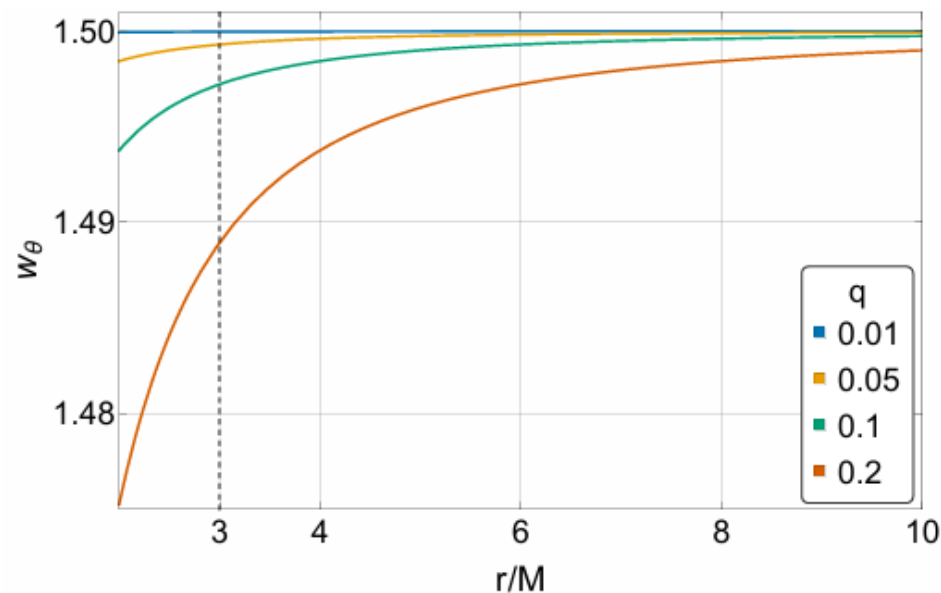
Examples: Bardeen Black hole

$$f(r) = 1/h(r) = 1 - \frac{2Mr^2}{(r^2 + q^2)^{3/2}}$$

$$\rho = -P_r = \frac{6Mq^2}{8\pi(r^2 + q^2)^{5/2}}$$

$$w_r = -1$$

$$w_\theta = \frac{3r^2 - 2q^2}{2(q^2 + r^2)}$$



$$\lambda \simeq \frac{1}{3\sqrt{3}M} \left(1 - \frac{q^2}{9M^2} \right)$$

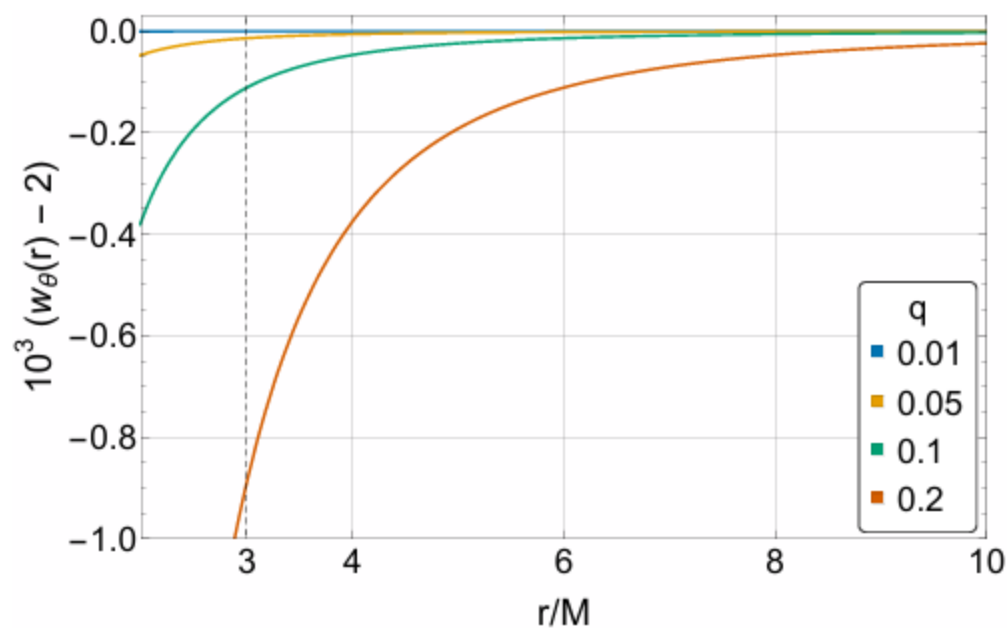
$$\Omega \simeq \frac{1}{3\sqrt{3}M} \left(1 + \frac{q^2}{6M^2} \right)$$

Examples: Hayward black hole

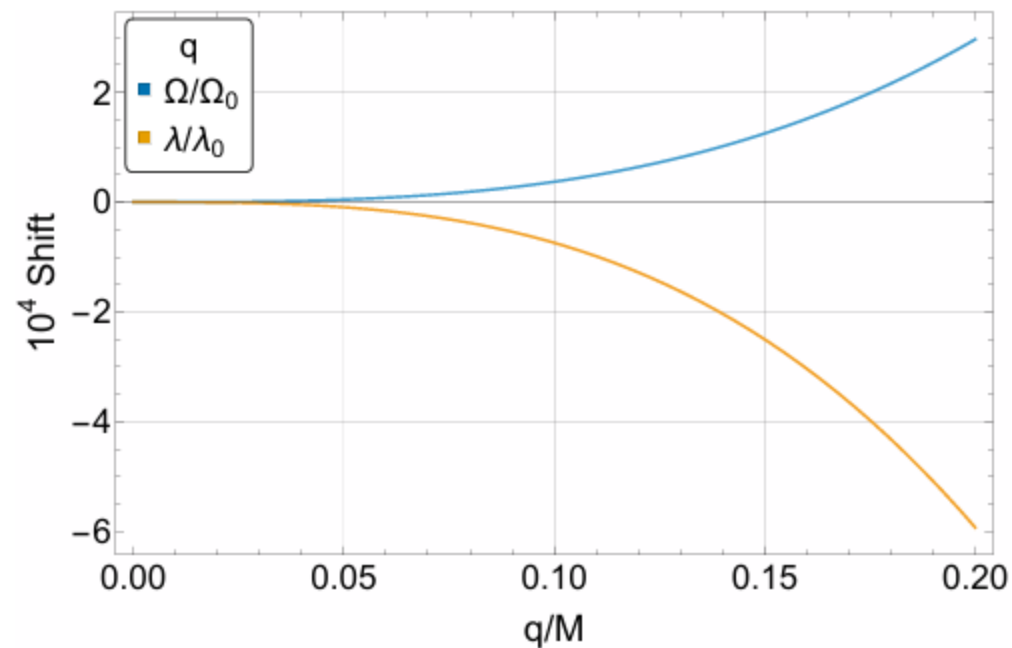
$$8\pi\rho = \frac{6q^3M}{(r^3 + q^3)^2}$$

$$w_r = -1$$

$$w_\theta = -\frac{q^3 - 2r^3}{q^3 + r^3}$$



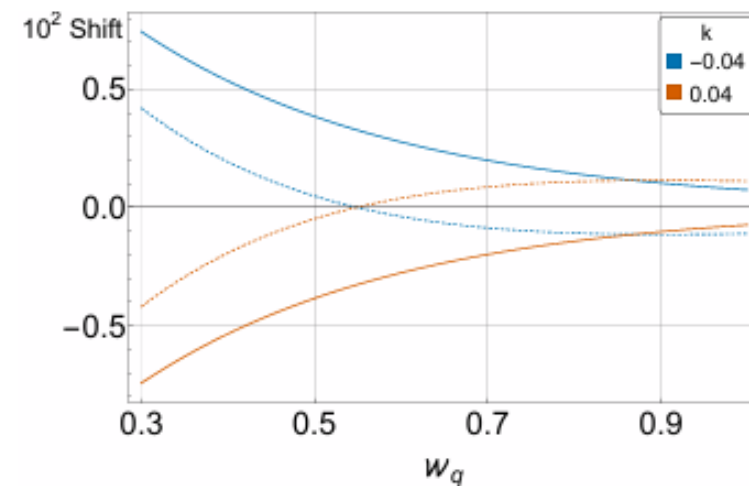
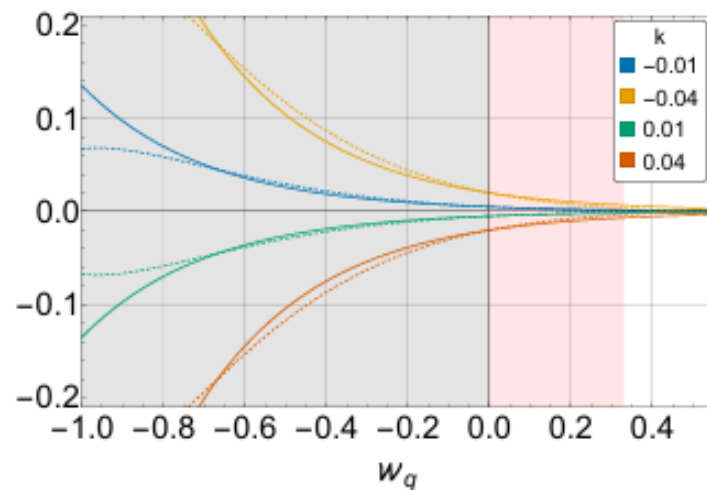
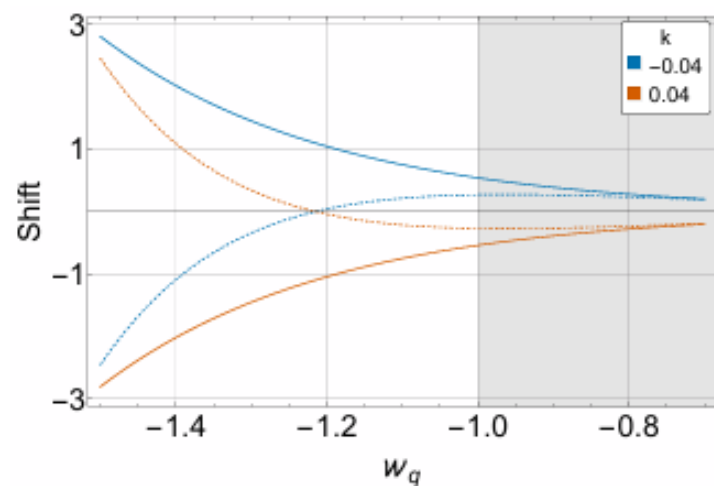
$$f(r) = 1/h(r) = 1 - \frac{2Mr^2}{r^3 + q^3}$$



$$\Omega \simeq \frac{1}{3\sqrt{3}M} \left(1 + \frac{q^3}{27M^3} \right)$$

$$\lambda \simeq \frac{1}{3\sqrt{3}M} \left(1 - \frac{2q^3}{27M^3} \right)$$

Examples: Kiselev black hole $f(r) = 1/h(r) = 1 - \frac{2M}{r} - \frac{K}{r^{1+3w_q}}$



$$\rho = -P_r = -\frac{3k w_q}{8\pi r^{3(1+w_q)}}$$

$$P_\theta = -\frac{3k w_q(1+3w_q)}{16\pi r^{3(1+w_q)}}$$

$$\Omega_\star \simeq \frac{1}{3\sqrt{3}M} \left(1 - \frac{3k}{2(3M)^{1+3w_q}} \right)$$

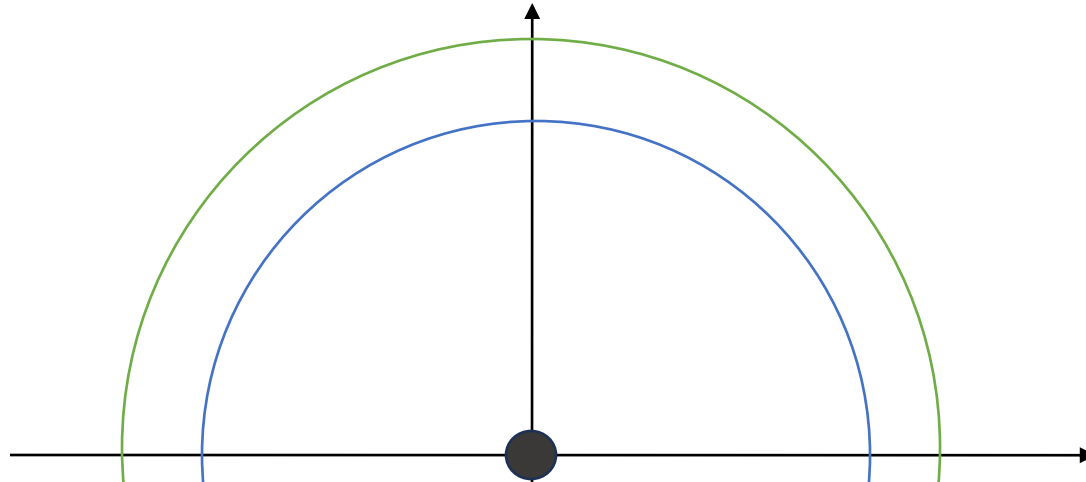
$$\lambda_\star \simeq \frac{1}{3\sqrt{3}M} \left(1 + \frac{3w_q(1+w_q) - 2}{4(3^{3w_q} M^{1+3w_q})} k \right)$$

From static to rotating

General metric

- Rotation is introduced through the Newman–Janis construction.
- We restrict to **equatorial motion**

$$\begin{aligned}
 ds^2 = & -f dt^2 - 2a \sin^2 \theta (1 - f) dt d\varphi + \{ \Sigma + (2 - f) a^2 \sin^2 \theta \} \sin^2 \theta d\varphi^2 \\
 & + \frac{\Sigma}{\Sigma f + a^2 \sin^2 \theta} dr^2 + \Sigma d\theta^2, \\
 f := & 1 - \frac{2m(r)r}{\Sigma}, \quad \Sigma := r^2 + a^2 \cos^2 \theta,
 \end{aligned}$$



From static to rotating

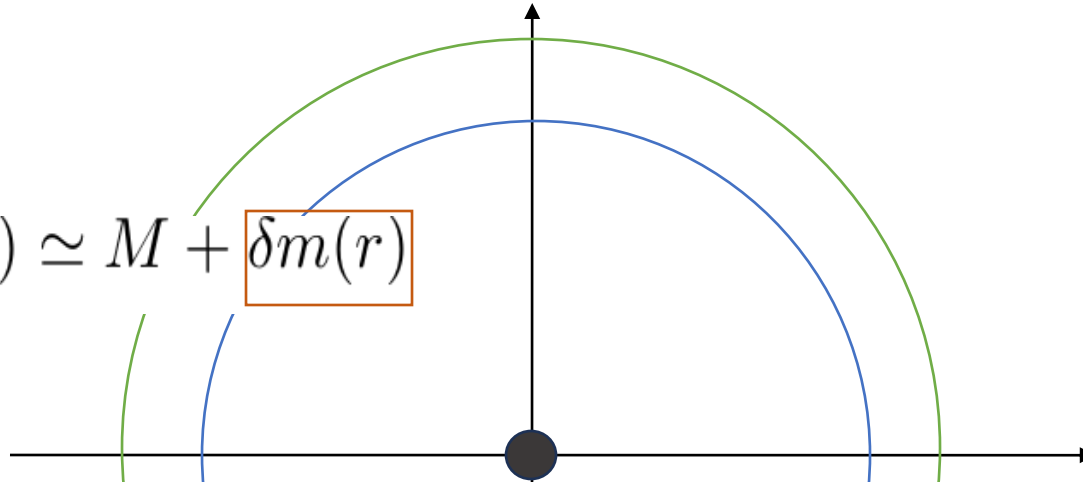
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 f := & 1 - \frac{2m(r)r}{\Sigma}, \quad \Sigma := r^2 + a^2 \cos^2 \theta,
 \end{aligned}$$

Kerr: $m(r) = M$

General: $m(r) \simeq M + \boxed{\delta m(r)}$



Spin splits co- and counter-rotating photon orbits

$$\omega_{\text{QNM}}^{(\varepsilon)} = \mathbf{m}\Omega_{\varepsilon} - i \left(n + \frac{1}{2} \right) \lambda_{\varepsilon} \quad \varepsilon = -1 \text{ (co-rotating)} \quad \varepsilon = +1 \text{ (counter-rotating)}$$

where $\mathbf{m} = -\varepsilon\ell$.

CO-ROTATING: $\varepsilon = -1$

**smaller Kerr photon-orbit
radius**

samples the source deeper in

COUNTER-ROTATING: $\varepsilon = +1$

**larger Kerr photon-orbit
radius**

**samples a different source
region**

Spin splits co- and counter-rotating photon orbits

$$\omega_{\text{QNM}}^{(\varepsilon)} = \mathbf{m}\Omega_\varepsilon - i \left(n + \frac{1}{2} \right) \lambda_\varepsilon \quad \varepsilon = -1 \text{ (co-rotating)} \quad \varepsilon = +1 \text{ (counter-rotating)}$$

where $\mathbf{m} = -\varepsilon \ell$ $\longrightarrow$ $\delta\hat{\Omega}_\varepsilon = -\varepsilon \delta\Omega_\varepsilon$

CO-ROTATING: $\varepsilon = -1$

**smaller Kerr photon-orbit
radius**

samples the source deeper in

COUNTER-ROTATING: $\varepsilon = +1$

**larger Kerr photon-orbit
radius**

**samples a different source
region**

Equatorial setup for the rotating geometry

Condition for circular orbits $V(r) = 0$ $V'(r) = 0$

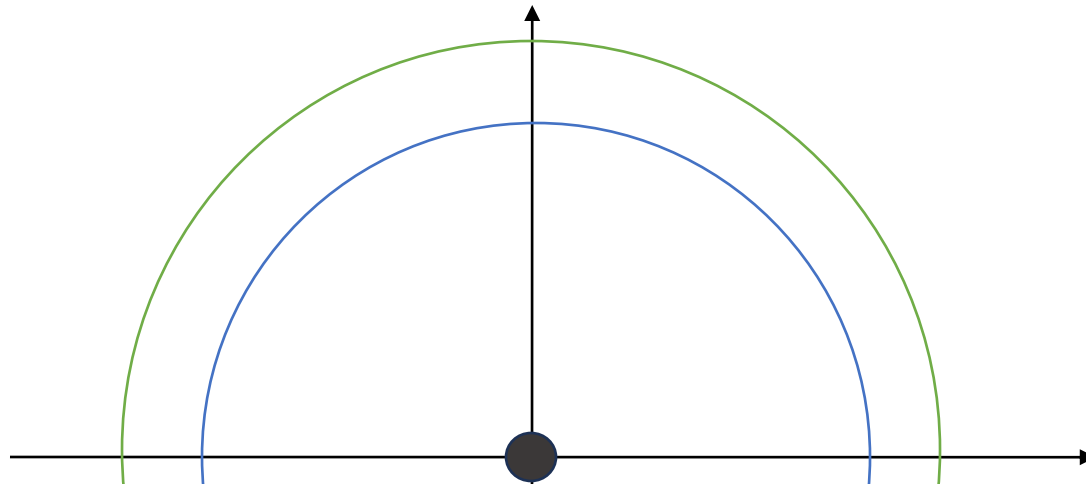
Then

$$\Omega := \frac{1}{D}$$

Orbital frequency

$$\lambda^2 := -\frac{V''}{\dot{t}^2}$$

Lyapunov exponent



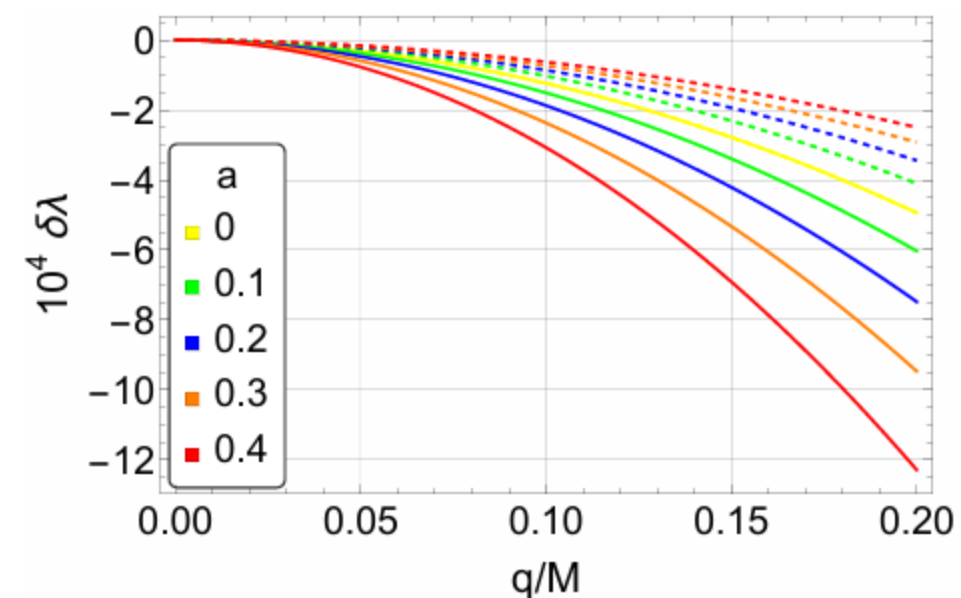
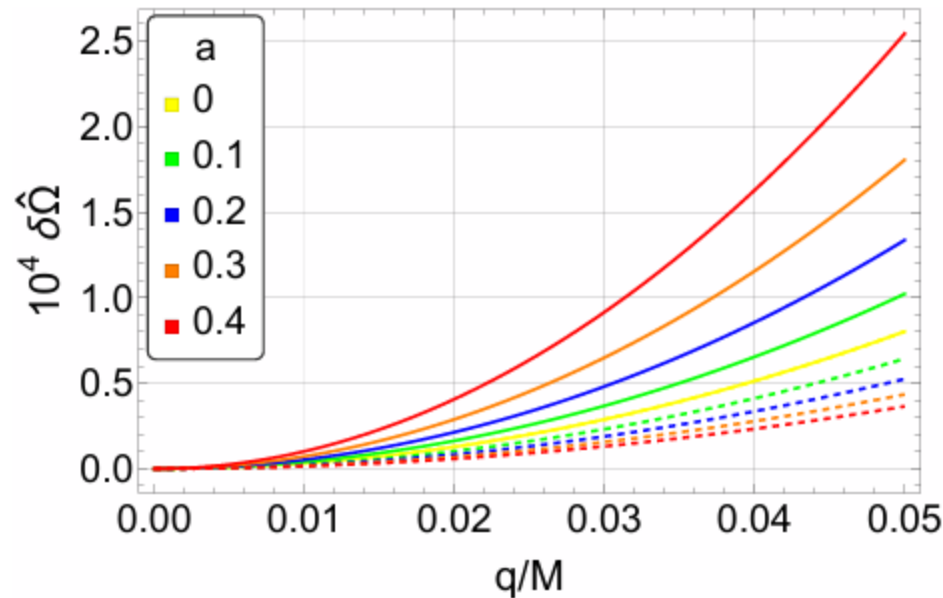
Rotation makes radial pressure explicit

$$\begin{aligned}\delta\hat{\Omega}_\epsilon &\longleftarrow \int_{r_{K,\epsilon}}^\infty s^2 w_r(s) \rho(s) ds, \\ \delta\lambda_\epsilon &\longleftarrow A_\epsilon \int_{r_{K,\epsilon}}^\infty s^2 w_r \rho ds + B_\epsilon r_{K,\epsilon}^2 w_r \rho + 2C_\epsilon r_{K,\epsilon}^2 w_\theta \rho\end{aligned}$$

where

$$A(r) \equiv \frac{2(r^3 + 13Mr^2 - 21M^2r - 9M^3)}{r(r-M)(3M+r)}, \quad B(r) \equiv 2(r-5M), \quad C(r) \equiv (r-M)r.$$

Spin separates two measurements of the same source



solid: counter-rotating · dashed: co-rotating

$a \rightarrow 0$
branches merge

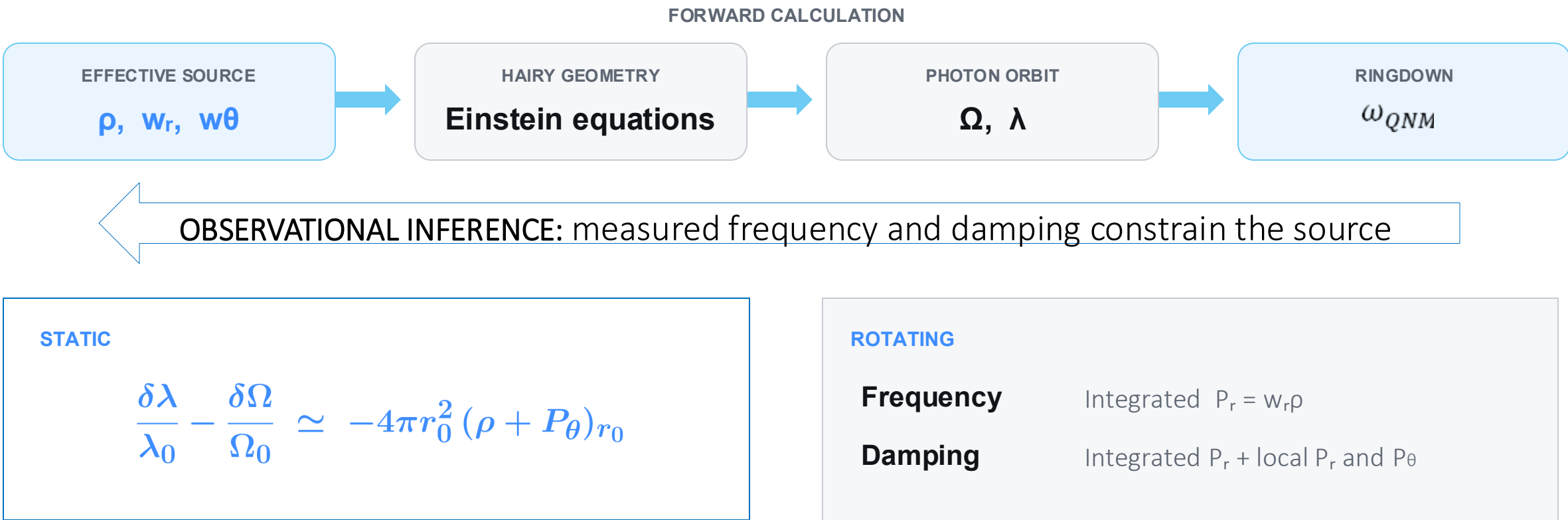
LARGER SPIN (a)
branch split grows

Rotation changes which source combinations enter the QNM shifts

	STATIC	ROTATING
ORBIT	one radius $r_0 = 3M$	two radii $r_{K,\epsilon}$
FREQUENCY	Integrated density $\int_r^\infty \rho(s) s^2 ds$	integrated radial pressure $\int_{r_{K,\epsilon}}^\infty s^2 w_r(s) \rho(s) ds,$
DAMPING	Integrated & local $\rho(1+w_\theta)$	integral & local $w_r \rho, w_\theta \rho$
KEY FEATURE	explicit w_r cancellation	w_r radial pressure appears explicitly

CONCLUSION

From the effective source to ringdown and back



Measurable ringdown shifts can constrain the effective source behind black-hole hair.

Thanks for your time